Automatic Inference for Value-Added Regressions*

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Abstract

A large empirical literature performs regressions of other variables on empirical Bayes shrinkage estimates, yet little is known about whether this approach leads to valid inference. In this paper, we consider a general class of value-added estimators and the properties of their corresponding regression coefficients. We demonstrate that this practice can lead to invalid inference if the shrinkage estimates neglect heteroskedasticity in the underlying noise. We show that properly constructed shrinkage estimates provide the solution, performing an automatic bias correction: the associated regression estimator is asymptotically unbiased, asymptotically normal, and efficient in the sense that it is asymptotically equivalent to regressing on the true (latent) value-added. Further, OLS standard errors from regressing on shrinkage estimates are consistent. As such, efficient inference is easy for practitioners to implement: simply regress outcomes on shrinkage estimates of value-added that account for the error structure.

`Key words:` shrinkage estimators, teacher value-added, error in variables

`JEL classification codes:` C12.

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1 Introduction

Empirical Bayes shrinkage estimators are widely applied when a large number of unit-specific parameters are present. In the canonical setting, noisy measurements of latent individual effects are observed. Empirical Bayes methods have been used to estimate teachers' value-added to student test scores (e.g. Chetty, Friedman, and Rockoff, 2014a,b), location effects (e.g. Chetty and Hendren, 2018), hospital quality (e.g. Hull, 2018), nursing home quality (e.g. Einav, Finkelstein, and Mahoney, 2025), firm-level discrimination (e.g. Kline, Rose, and Walters, 2022), among others. In many of these settings, the shrinkage estimates are not the final object of interest but are instead used as inputs to downstream analyses.

In empirical work, researchers increasingly use shrinkage estimates as downstream regressors to study the relationship between latent individual effects and long-term outcomes. Most studies employ individualized shrinkage, where the shrinkage strategies vary across units to reflect heterogeneity in measurement precision. Examples include Jacob and Lefgren (2007), Jacob and Lefgren (2008), Kane and Staiger (2008), Chandra, Finkelstein, Sacarny, and Syverson (2016), Jackson (2018), Abdulkadiroğlu, Pathak, Schellenberg, and Walters (2020), Bau and Das (2020), Biasi and Sarsons (2022), Warnick, Light, and Yim (2024), Andrabi, Bau, Das, and Khwaja (2025), and Angelova, Dobbie, and Yang (2025). In these studies, individualized shrinkage is motivated by the oracle posterior mean in parametric empirical Bayes methods. Unlike equal shrinkage followed by, e.g. Chetty et al. (2014a,b), individualized shrinkage explicitly models and estimates heterogeneity in measurement precision across units. In practice, researchers often treat individualized shrinkage estimates as if they were true effects when using them as regressors and report conventional standard errors. However, the impact of different forms of shrinkage on the validity and efficiency of downstream inference remains unclear.

This paper is the first to formalize the impact of different shrinkage schemes on the properties of regressions that use individualized shrinkage estimators as regressors, and to explain when such regressions do and do not provide automatic inference. We establish conditions under which individualized shrinkage yields asymptotically valid inference when used properly as a downstream regressor. The resulting downstream coefficient estimator is efficient, being asymptotically equivalent to the infeasible OLS regression on the true latent effects. This improvement is not immediate. Many empirical implementations model measurement precision solely by the number of measurements per individual, although idiosyncratic noise variances may also differ across individuals. We show that accounting only for heterogeneity in the number of measurements can yield invalid inference if noise variances are correlated with the number of measurements, though inference remains valid when the two are independent. Incorporating both sources of heterogeneity restores efficiency and ensures robustness to such dependence. In our on-going research, we also extend the method to a new estimator that applies under the important case with dependence of individual effects on the variance of individual means.

We focus on linear shrinkage estimators due to their popularity and ease of implementation. Typical linear shrinkage estimators are constructed as a weighted average of the individual mean and the pooled mean, with estimated weights reflecting the signal-to-noise ratio. These estimators can be classified as common-weight (equal shrinkage) or individual-weight (individualized shrinkage). They are the empirical Bayes posterior means under normality of both noise and individual effects (Efron and Morris, 1973; Morris, 1983). In general, the purpose of recovering the posterior means and the purpose of downstream regressions are different. For instance, common-weight shrinkage does not align with the intuition that components measured with greater noise should be shrunk more heavily (Efron and Morris, 1973; Morris, 1983; Xie, Kou, and Brown, 2012), yet it performs well for downstream regression, as we will show. Second, the normality assumptions are unnecessary for valid inference in downstream regressions. Finally, one might be concerned that inference suffers from a generated regressor problem (Pagan, 1984). It is therefore not obvious whether individual-weight shrinkage can ensure valid plug-in inference when used as downstream regressors.

To formalize our analysis, we first restate the established baseline result that the unshrunk individual mean suffers from classical errors-in-variables attenuation bias and yields invalid inference when plugged in as a downstream regressor. We then move to our main results on individual-weight shrinkage. We first delineate the precise conditions under which inference can fail, even when heterogeneity in the number of measurements is accounted for. We then establish how a correctly specified, fully individualized shrinkage estimator restores asymptotic efficiency and inferential validity. We extend the results to shrinkage with dependence between individual effects and the variance of individual means, and show that it has the same asymptotic performance as using the true latent individual effects as the regressor. Finally, as a point of comparison, we analyze the common-weight estimator. We show that because of its close connection to conventional bias-correction methods, this estimator also addresses the inference problem, providing a useful benchmark for our primary results on the more empirically prevalent individual-weight approach.

We illustrate the method using the firm discrimination data of Kline et al. (2022) and the school value-added data of Andrabi et al. (2025). In the first application, individual-weight heteroskedastic shrinkage yields well-behaved confidence intervals when used as regressors to predict future discrimination levels. In the second application, individual-weight heteroskedastic shrinkage yields slightly different confidence intervals when used as regressors to predict private school fees, and it reaffirms the positive coefficient of school value-added in predicting private school fees.

As practical take-aways, implementation of the shrinkage estimator is straightforward. To illustrate, consider the teacher value-added application. We estimate individual teacher value-added using a shrinkage estimator formed as a weighted average of each teacher's individual mean score and all teachers' pooled mean score. The weights vary across teachers and are determined by the signal-to-noise ratio, which is estimated by within- and across-teacher variance components. In predicting long-run outcomes as a function of latent value-added, we simply regress the outcomes on these shrinkage estimates. From the reported results, conventional OLS standard errors and confidence intervals are valid for inference.

Related Literature The use of shrinkage estimators as downstream regressors started from common-weight shrinkage. Regressing on common-weight shrinkage estimators can be traced back to Whittemore (1989). They demonstrate via simulations that common-weight shrinkage yields consistent downstream coefficients, and point out its equivalence to bias-correction methods. Building on this insight, Guo and Ghosh (2012) provide theoretical justifications regarding the quadratic risk reduction of such estimators for downstream coefficients. Along similar lines within common-weight shrinkage, Chetty et al. (2014a,b) construct shrinkage estimators based on the best linear predictor, emphasizing that the estimator is bias-free when used as the regressor. Their estimator can be viewed as equivalent to IV methods, and we unify this IV perspective along with the bias-correction perspective of common-weight shrinkage in Section 3.5. Deeb (2021) further exploits the equivalence to IV methods and develops inference results for Chetty et al. (2014a,b)'s estimator, highlighting the need to adjust conventional OLS standard errors to account for errors in estimation of individual effects and nuisance parameters. While this line of work clarifies inference in the common-weight shrinkage setting, it essentially relies on the equivalence to IV and does not extend to individual-weight shrinkage. Given the prevalence of individual-weight shrinkage, we depart from this equivalence-based view and establish a distinct framework.

This paper is therefore also closely related to the literature on individual-weight shrinkage estimators as plug-in regressors. Unlike the common-weight approach which applies a uniform adjustment factor to correct attenuation bias, the individual-weight approach uses unit-specific adjustments, improving MSE but introducing complications. Jacob and Lefgren (2007), Kane and Staiger (2008), Abdulkadiroğlu et al. (2020), Walters (2024), and Andrabi et al. (2025) briefly discuss the consistency of regression estimators based on individual-weight shrinkage. In particular, Walters (2024) highlights independence between individual effects and the variance of the noise as a necessary condition. Beyond such consistency results, there is no general framework establishing inference and the validity of standard errors for individual-weight plug-in regressions. We instead develop an asymptotic framework that delivers conditions under which the conventional OLS inference reported in applications are valid, thereby justifying standard empirical practice.

Beyond shrinkage estimators, there have been discussions about non-shrinkage options for the single purpose of downstream models. Suitable for nonlinear downstream models, Chang, Huang, Chen, and Liao (2024) leverage information from the estimated prior to develop correction methods. For downstream regressions, this strand is largely bias-correction. Since common-weight shrinkage has equivalence to bias-correction methods as discussed above, they can also be viewed as in the same category. Bonhomme and Denis (2024) emphasize correct estimation of moments for value-added which naturally yields correct regression estimates, and provide general solutions applicable across various cases. Chen, Gu, and Kwon (2025) emphasize the robustness of the classical bias-correction method for downstream coefficients, showing that it remains consistent even when individual effects are correlated with the variance of the noise. We establish conditions under which simpler conventional shrinkage estimators perform well for downstream regressions in the absence of such dependence, and then extend to the case with such dependence in the on-going research.

Outline This paper proceeds as follows. Section 2 introduces the framework and its implications. Section 3 discusses a rich form of shrinkage estimators, establishes the theoretical properties, and provides implementation guidelines. Section 4 reports Monte Carlo simulations illustrating the finite-sample properties of the estimators. Section 5 applies the method to labor market discrimination, and Section 6 applies it to school value-added, and Section 7 concludes.

2 Setup

2.1 Model

In empirical Bayes applications, each unit i is associated with a latent value-added parameter θ_i. Let $\boldsymbol{\theta} \coloneqq (\theta_i)_{i=1}^n$ denote the vector of these latent effects for the n sampled units. People are interested in the downstream effect of value-added θ_i on the outcome Y_i. The downstream regression model takes the form

$$Y_i = \alpha + \beta\theta_i + u_i, \tag{1}$$

where the coefficient β captures the downstream effect. The error term u_i has mean zero and is uncorrelated with θ_i, $\mathbb{E}\left[u_i\theta_i\right] = 0$. In the motivating example of teacher value-added, θ_i stands for teacher i's latent value-added to students' test scores, and Y_i is the average long-term outcome of students taught by teacher i. One may be interested in how the effect on short-term test scores translates to long-term outcomes—such as college attendance or earnings—through the regression inference on β.

Inference on β faces the challenge that $\boldsymbol{\theta}$ is unobserved. Instead, for each unit i we only have data $(Y_i, X_i) \coloneqq (Y_i, X_{i,1}, ..., X_{i,J_i})$, $i = 1, ..., n$, where Y_i is the outcome and $X_{i,j}$ is the measurement. Specifically, $X_i \in \mathbb{R}^{J_i}$ are noisy repeated measurements for θ_i from the model

$$X_{i,j} = \theta_i + \epsilon_{i,j}, \tag{2}$$

where $\epsilon_{i,j}$ is the noise term. Within the example, $X_{i,j}$ is the short-term test score outcome of student j taught by teacher i. The individual mean score is denoted by $\bar{X}_i$

and the pooled mean score by $\bar{X}$. In this paper, we primarily focus on the case where the noise is independent of the value-added, i.e. $\epsilon_{i,j} \perp\!\!\!\perp \theta_i$. We also assume the noise is independent of the structural error u_i, i.e. $\epsilon_{i,j} \perp\!\!\!\perp u_i$, which ensures identification of β. Before proceeding, we discuss how the observation-specific sample size J_i relates to n.

2.2 Asymptotic Framework

As is standard, we will use empirical Bayes or shrinkage methods to estimate $\boldsymbol{\theta}$. In this method, it is commonly assumed that $\bar{X}_i \mid \theta_i \sim N(\theta_i, \sigma_i^2/J_i)$. Without distributional knowledge about the noise, the normality of $\bar{X}_i$ is typically justified by the central limit theorem with J_i reasonably large (Walters, 2024). It therefore makes sense to adopt a framework where both J_i and n are "large" in a suitable sense. In the teacher value-added example, J_i often has a similar magnitude to $\sqrt{n}$, with about tens of students versus thousands of teachers. For example, in the study of North Carolina data in Deeb (2021), the total number of teachers $n = 5266$, and the total number of students is 388,191, so we would expect J_i is on average about 74, which is comparable to $\sqrt{n} \approx 73$. In another example from Bau and Das (2020), $\sqrt{n} \approx 39$ and J_i is on average about 15. Since J_i^{-1} is proportional to the measurement error of θ_i, and so is $n^{-1/2}$ to the sampling error for inference on β in (1), we would like to model the framework so that both are of a similar magnitude. In addition, in finite samples there is error in both estimating $\boldsymbol{\theta}$ and β. This framework keeps both alive asymptotically, to mimic the finite-sample problem where both play a role.

To formalize the idea, we consider the following asymptotic framework. As $n \to +\infty$, the DGP of the other variables are fixed, but for J_i we shall assume

$$\sqrt{n}\mathbb{E}\left[\frac{1}{J_i}\right] \to \kappa \in [0, +\infty), \tag{3}$$

$$\sqrt{n}\mathbb{E}\left[\frac{1}{J_i^2}\right] \to 0. \tag{4}$$

Note that if we have the same number of measurements, $J_i = J$, then (4) is implied by (3). Thus (3) is the essential condition, under which the asymptotic problem minics the finite sample problem. The product $\sqrt{n}\mathbb{E}\left[J_i^{-1}\right]$ quantifies the relative scale of measurement error to sampling error. When $\kappa = 0$, J_i grows asymptotically much larger than $\sqrt{n}$, implying that measurement error is negligible for the purpose of inference in the regression. When $\kappa > 0$, measurement error is of a similar order to the sampling error. In the previous two application examples, we have $\hat{\kappa}_1 \approx 73/74 \approx 1$, and $\hat{\kappa}_2 \approx 39/15 \approx 2.6$. The approximation is a lower bound. Jensen's inequality produces a larger value if J_i varies with i. A finite positive κ thus seems appropriate for the context in which tens of students and thousands of teachers coexist.

Our asymptotic framework is s related to the small-variance approximation in Chesher (1991) and the limit argument in Evdokimov and Zeleneev (2019, 2023, 2024); Battaglia, Christensen, Hansen, and Sacher (2024). The setup and conditions are also analogous to the factor-augmented regression literature (e.g. Gonçalves and Perron, 2014).

As we will show in Section 3.1, this asymptotic framework implies that simply regressing Y_i on $\bar{X}_i$ without further adjustment yields invalid inference.

The primary implication of this framework is that the measurement error in $\bar{X}_i$ does not vanish relative to the sampling error for β. This persistence of measurement error is the central econometric challenge we address. As we will show in Section 3.1, this asymptotic setup confirms that the naive unshrunk estimator (simply regressing Y_i on $\bar{X}_i$) suffers from classical errors-in-variables bias and yields invalid inference. This failure provides the necessary baseline and motivation for the rest of the paper.

3 Shrinkage Estimators as Regressors

In this section, we analyze the inferential properties of different shrinkage estimators when used as downstream regressors. We follow the standard two-step workflow in empirical work: Step 1: Estimate the value-added $\boldsymbol{\theta}$ with $\hat{\boldsymbol{\theta}}$; Step 2: Perform inference on β by regressing Y_i on $\hat{\theta}_i$. Our interest is on the validity of inference in Step 2 when different shrinkage estimators are used in Step 1.

We consider four primary estimators for $\hat{\boldsymbol{\theta}}$, which we analyze in the order of our main results. We begin with the unshrunk fixed-effect (FE) estimator as a baseline and then move to individual-weight shrinkage. We explicitly distinguish between two forms: homoskedastic individual-weight shrinkage (HO), which models heterogeneity in measurement precision solely by the number of measurements (J_i), and heteroskedastic individual-weight shrinkage (HE), which incorporates both sources of heterogeneity by also modeling idiosyncratic noise variances. Finally, as a benchmark, we analyse common-weight shrinkage (CW), which corresponds to equal shrinkage approaches and applies a single, uniform weight. We discuss the performance of each for the downstream inferential task.

3.1 Fixed Effects (FE)

As the baseline, we discuss using the fixed-effect estimator (FE) with no shrinkage, i.e. the sample average $\bar{\mathbf{X}} \coloneqq (\bar{X}_1, ..., \bar{X}_n)$ as $\hat{\boldsymbol{\theta}}_{\mathrm{FE}}$. We now analyze the performance of this estimator for the Step 2 downstream regression.

Using $\bar{X}_i$ as a regressor suffers from the classical errors-in-variable (EIV) problem, leading to attenuation bias. In our setting, the measurement error is of the same order as the sampling error, kept small but alive. We are under the setting with (3) and (4). First, we state and discuss other assumptions that are needed for the asymptotic properties.

Assumption 3.1. *1. J_i is independent of θ_i, u_i and $\epsilon_{i,j}$, and $J_i \geq 3, a.s.$*

2. $\mathbb{E}[u_i] = 0$, $\mathbb{E}(u_i\theta_i) = 0$. Y_i has finite fourth moments.

3. $\mathbb{E}[\epsilon_{i,j} \mid \sigma_i^2] = 0$, $\mathbb{E}\left[\epsilon_{i,j}^2 \mid \sigma_i^2\right] = \sigma_i^2$, $\mathbb{E}\left[|\epsilon_{i,j}|^L \mid \sigma_i^2\right] \leq K\sigma_i^L$, for some $L \geq 3$. Also, $\epsilon_{i,j} \perp\!\!\!\perp \theta_i$.

4. $u_i \perp\!\!\!\perp \epsilon_{i,j} \big| \theta_i$.

5. θ_i has finite fourth moments.

6. σ_i^2 has finite eighth moments.

For the moment, we assume independent numbers of measurements J_i in Assumption 3.1.1. In Appendix D, the condition will be relaxed and allow dependence between J_i and σ_i^2, accompanied by slightly stronger assumptions than (3) and (4). We keep the independence for the ease of exposition. Assumption 3.1.2 accommodates heteroskedasticity in the OLS regression error u_i, and allows the asymptotic normality to hold if the true θ_i is observed.

Assumption 3.1.3 places moment conditions on the noise $\epsilon_{i,j}$. The moment conditions are more general and cover a wide range of distributions for $\epsilon_{i,j}/\sigma_i$, including normal, bounded, and even some asymmetric distributions. The following independence $\epsilon_{i,j} \perp\!\!\!\perp \theta_i$ is standard in empirical Bayes literature to justify shrinkage estimators without covariates. Assumption 3.1.4 ensures identification of β. Assumption 3.1.5

and Assumption 3.1.6 are standard asymptotic conditions that are possible to be relaxed with more technical arguments.

Denote by $\hat{\beta}_{\text{FE}}$ the OLS estimator from Y_i regressed on $\hat{\theta}_{i,\text{FE}}$. The problem of invalid inference is established by the following result.

Proposition 3.1. *Suppose the asymptotic framework satisfies* (3) *and* (4)*. Under Assumption 3.1, we have*

$$\sqrt{n}\left(\hat{\beta}_{\text{FE}}-\beta\right)\to_d N\left(-\kappa\beta\frac{\mathbb{E}\left[\sigma_i^2\right]}{\text{Var}\left(\theta_i\right)},\frac{\mathbb{E}\left[u_i^2\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^2\right]}{\left(\text{Var}\left(\theta_i\right)\right)^2}\right).$$

The proposition's result holds even without (4). In the special case where $\kappa=0$ (meaning measurement error is asymptotically negligible), the bias term disappears, the asymptotic distribution of $\sqrt{n}\left(\hat{\beta}_{\text{FE}}-\beta\right)$ is centered at zero, and standard inference would be valid. In our primary framework with $\kappa>0$, $\hat{\beta}_{\text{FE}}$ remains consistent and attains the efficient asymptotic variance. However, its asymptotic distribution is centered at a biased value due to the attenuation EIV problem. Supposing the standard regression output reports the standard error $\text{SE}(\hat{\beta}_{\text{FE}})$ calculated in the usual way, a confidence interval constructed in the following (e.g., $\hat{\beta}_{\text{FE}}\pm 1.96\times\text{SE}(\hat{\beta}_{\text{FE}})$) will fail to cover the true β at the nominal rate. Therefore, despite accommodating large J_i, the persistent measurement error renders conventional OLS inference misleading for $\hat{\theta}_{i,\text{FE}}$.

As a result, the unshrunk $\hat{\boldsymbol{\theta}}_{\text{FE}}$ fails to provide valid plug-in inference in Step 2. This failure of the baseline estimator motivates the need for the shrinkage methods we analyse next.

3.2 Homoskedastic Individual-Weight Shrinkage (HO)

In this subsection, we discuss the individual-weight shrinkage when the variance of $\bar{X}_i$ is believed to only depend on the number of measurements J_i, i.e. $\text{Var}\left(\bar{X}_i\mid\theta_i\right)=\sigma^2/J_i$. We refer to this case as homoskedastic individual-weight shrinkage (HO), since $\epsilon_{i,j}$ is homoskedastic across i. Here the primary source of heteroskedasticity is believed to arise from variation in the number of measurements, which must therefore be taken into account.

Following the standard implementation of the method in, e.g., Jacob and Lefgren

(2007) and Kane and Staiger (2008), the estimator of $\boldsymbol{\theta}$ is constructed as

$$\hat{\theta}_{i,\mathrm{HO}} := \frac{\hat{\sigma}_\theta^2}{\frac{1}{J_i}\hat{\sigma}^2 + \hat{\sigma}_\theta^2}\bar{X}_i + \frac{\frac{1}{J_i}\hat{\sigma}^2}{\frac{1}{J_i}\hat{\sigma}^2 + \hat{\sigma}_\theta^2}\bar{X},$$

where

$$\hat{\sigma}^2 := \widehat{\mathrm{Var}}\left(X_{i,j} - \bar{X}_i\right)$$
$$\hat{\sigma}_\theta^2 := \widehat{\mathrm{Cov}}\left(\bar{X}_{i,t}, \bar{X}_{i,t-1}\right).$$

The estimator is the empirical Bayes posterior mean under normality of θ_i and $\epsilon_{i,j}$. The weight $w_{i,\mathrm{HO}}$ is obtained by replacing the oracle shrinkage weight

$$\frac{\mathrm{Var}(\theta_i)}{\mathrm{Var}(X_{i,j} \mid \theta_i)/J_i + \mathrm{Var}(\theta_i)}$$

with its empirical counterparts.

Since our setting does not involve time periods, $\bar{X}_{i,t}$, $\bar{X}_{i,t-1}$ can be interpreted as splitting the measurements for each i into two subsets and computing their averages. Note that $\hat{\sigma}^2$ is applied uniformly across units, with variation in J_i reflecting differences in measurement precision.

If homoskedasticity holds, this method is expected to correctly model the signal-to-noise ratio. Indeed, as we show below in Proposition 3.2, homoskedasticity ensures that regressing Y_i on $\hat{\theta}_{i,\mathrm{HO}}$ yields asymptotic unbiasedness and valid inference on β.

If the true DGP is heteroskedastic, however, there would be problems in inference when homoskedastic weights are applied. Denoting the variance of $\bar{X}_i \mid \theta_i$ by σ_i^2/J_i, regressing Y_i on $\hat{\theta}_{i,\mathrm{HO}}$ will lead to invalid inference on β in some heteroskedastic cases. A leading case is when the number of measurements J_i is correlated with the variance σ_i^2. In that case, it is natural to have J_i endogenously increased for observations with large σ_i^2, and then the downstream regression generates asymptotically biased OLS estimates. The following result substantiates this claim, with details and proofs in Appendix D.

Proposition 3.2. *Suppose the asymptotic framework satisfies* $n^{3/2}\mathbb{E}\left[\frac{1}{J_i^3}\right] \to \kappa^3$. *If we allow correlation between* J_i *and* σ_i^2, *then under Assumption D.1 and D.2, there* *exist DGPs with $\gamma > 0$ in which*

$$\sqrt{n}\left(\hat{\beta}_{HO} - \beta\right) \to_d N(\gamma\beta, \frac{\mathbb{E}\left[u_i^2\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)^2\right]}{\left(\mathrm{Var}\left(\theta_i\right)\right)^2}).$$

If instead J_i and σ_i^2 are independent, then

$$\sqrt{n}\left(\hat{\beta}_{HO} - \beta\right) \to_d N(0, \frac{\mathbb{E}\left[u_i^2\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)^2\right]}{\left(\mathrm{Var}\left(\theta_i\right)\right)^2}).$$

Here, the condition $n^{3/2}\mathbb{E}\left[\frac{1}{J_i^3}\right] \to \kappa^3$ is generally stronger than (3) and (4), though only mildly so. They are equivalent if we have constant J_i. When homoskedasticity holds, independence holds and we have the asymptotic unbiasedness discussed above.

The result in Proposition 3.2 clarifies the conditions required for the HO estimator to yield valid inference. While inference is valid if the homoskedasticity assumption holds or if J_i and σ_i^2 are independent, the estimator's performance is sensitive to violations of these conditions. As $\gamma > 0$ in the bias term, the asymptotic bias is amplification instead of classical EIV attenuation bias. Generally, the sign of the bias depends on the dependence of J_i and σ_i^2. It motivates the heteroskedastic estimator we consider next, which is designed to be robust to such dependence.

3.3 Heteroskedastic Individual-Weight Shrinkage (HE)

In this subsection, we discuss the heteroskedastic individual-weight (HE) estimator. This estimator directly addresses the sensitivity of the HO estimator by accommodating heteroskedasticity in the measurement error. It is the fully individualized shrinkage estimator in that it accounts for both sources of heterogeneity: the number of measurements J_i and the idiosyncratic noise variance σ_i^2.

The heteroskedastic individual-weight shrinkage estimator $\hat{\boldsymbol{\theta}}_{\mathrm{HE}}$ is defined as $\hat{\boldsymbol{\theta}}_{\mathrm{HE}} := \left(\hat{\theta}_{i,\mathrm{HE}}\right)_{i=1}^{n}$, with

$$\begin{aligned} \hat{\theta}_{i,\mathrm{HE}} &:= w_{i,\mathrm{HE}}\bar{X}_i + (1 - w_{i,\mathrm{HE}})\bar{X}, \text{ where} \\ w_{i,\mathrm{HE}} &:= \frac{\hat{V}}{\frac{1}{J_i}\hat{\sigma}_i^2 + \hat{V}}. \end{aligned} \tag{5}$$

Here, $\hat{V}$ is the estimator of $\operatorname{Var}(\theta_i)$, and $\hat{\sigma}_i^2$ is the estimator of $\operatorname{Var}(X_{i,j} \mid \theta_i)$. The variance estimators are constructed following Kline, Saggio, and Sølvsten (2020) and Kline et al. (2022):

$$\hat{\sigma}_i^2 \coloneqq \frac{1}{J_i - 1} \sum_{j=1}^{J_i} \left(X_{i,j} - \bar{X}_i\right)^2,$$
$$\hat{V} \coloneqq \frac{1}{n} \sum_{k=1}^{n} \left(\bar{X}_k - \bar{X}\right)^2 - \frac{n-1}{n^2} \sum_{k=1}^{n} \frac{1}{J_k} \hat{\sigma}_k^2.$$

As an extension of the HO estimator, the weights $w_{i,\mathrm{HE}}$ accommodate heteroskedastic measurement error by allowing $\operatorname{Var}(X_{i,j} \mid \theta_i)$ to differ across units, thereby ensuring robustness to heteroskedasticity.

We take an intermediate step before deriving the properties of $\hat{\beta}_{\mathrm{HE}}$, the downstream OLS estimator of regressing Y_i on $\hat{\theta}_{i,\mathrm{HE}}$. Since the estimator $\hat{\theta}_{i,\mathrm{HE}}$ is an empirical Bayes estimator, with the estimated prior variance $\hat{V}$ and prior mean $\bar{X}$ for a normal prior on θ_i. As a preparation, we start by studying the property of OLS estimator $\hat{\beta}_{c,\mathrm{HE}}$ regressing Y_i on the semi-oracle estimator $\hat{\theta}_{i,c,\mathrm{HE}}$ (with prior mean still estimated from the data),

$$\hat{\theta}_{i,c,\mathrm{HE}} \coloneqq c_i \bar{X}_i + (1 - c_i)\, \bar{X}, \text{ where}$$
$$c_i \coloneqq \frac{\operatorname{Var}(\theta_i)}{\frac{1}{J_i}\hat{\sigma}_i^2 + \operatorname{Var}(\theta_i)}.$$

Lemma B.3 shows that the estimated variance $\hat{V}$ is a consistent estimator for the true variance $\operatorname{Var}(\theta_i)$. Thus, we build the properties of $\hat{\beta}_{\mathrm{HE}}$ based on those of $\hat{\beta}_{c,\mathrm{HE}}$. The following result, which is of independent interest, shows that regressing Y_i on the semi-oracle estimator $\hat{\theta}_{i,c,\mathrm{HE}}$ leads to an unbiased asymptotic distribution with efficient variance.

Lemma 3.1. *Suppose the asymptotic framework satisfies* (3) *and* (4)*. Then under Assumption 3.1 we have*

$$\sqrt{n}\left(\hat{\beta}_{c,\mathrm{HE}} - \beta\right) \to_d N\left(0, \frac{\mathbb{E}\left[u_i^2 \left(\theta_i - \mathbb{E}[\theta_i]\right)^2\right]}{\left(\operatorname{Var}(\theta_i)\right)^2}\right).$$

Lemma 3.1 shows that regressing Y_i on $\hat{\theta}_{i,c,\mathrm{HE}}$ leads to an asymptotically unbiased estimator of β in stylized cases where $\mathrm{Var}(\theta_i)$ is known. The asymptotic distribution is the same as the optimal one if the true θ_i is observed. Since $\hat{\boldsymbol{\theta}}_{c,\mathrm{HE}}$ is infeasible, we next study the properties of the feasible shrinkage estimator $\hat{\boldsymbol{\theta}}_{\mathrm{HE}}$.

The following result builds on Lemma 3.1 and establishes the asymptotic distribution if we use the feasible shrinkage estimator $\hat{\boldsymbol{\theta}}_{\mathrm{HE}}$.

Theorem 3.3. *Suppose the asymptotic framework satisfies* (3) *and* (4). *Then under Assumption 3.1, we have that* $\hat{\beta}_{\mathrm{HE}}$ *and* $\hat{\beta}_{c,\mathrm{HE}}$ *are first-order asymptotically equivalent:*

$$\sqrt{n}\left(\hat{\beta}_{\mathrm{HE}}-\beta\right)=\sqrt{n}\left(\hat{\beta}_{c,\mathrm{HE}}-\beta\right)+o_p(1),$$

and so

$$\sqrt{n}\left(\hat{\beta}_{\mathrm{HE}}-\beta\right)\to_d N\left(0,\frac{\mathbb{E}\left[u_i^2\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^2\right]}{\left(\mathrm{Var}\left(\theta_i\right)\right)^2}\right).$$

Theorem 3.3 establishes that the HE estimator $\hat{\boldsymbol{\theta}}_{\mathrm{HE}}$ performs asymptotically equivalent to knowing the true variance $\mathrm{Var}(\theta_i)$. Also, regression on $\hat{\theta}_{i,\mathrm{HE}}$ reaches the semiparametric efficiency bound (See the unconditional moment case of Chamberlain, 1987) and is as good as knowing the true θ_i. Even though it is assumed that $\sqrt{n}\mathbb{E}\left[J_i^{-1}\right]\to\kappa$, the ratio κ—representing the ratio of measurement error to sampling error—eventually doesn't show up in the asymptotic distribution.

Recall from Proposition 3.1 that the unshrunk FE estimator's asymptotic distribution was biased by the κ term, leading to invalid inference. Here, even though we remain in the same $\kappa>0$ regime where measurement error is persistent, that bias term is absent. The HE shrinkage procedure perfectly corrects for the EIV problem, and the inferential issue from the baseline FE case is automatically resolved.

In Section 3.2, we showed that under heteroskedasticity, HO yields invalid inference for β. By contrast, as we show now, HE delivers an asymptotically unbiased estimator of β under the same conditions. In this sense, HE provides a more robust approach, and the following proposition summarizes the result. Details and proofs are given in Appendix D.

Proposition 3.4. *Suppose the asymptotic framework satisfies* $n^{3/2}\mathbb{E}\left[\frac{1}{J_i^3}\right]\to\kappa^3$. *With* *any dependence between J_i and σ_i^2, under Assumption D.1 and D.2 we have*

$$\sqrt{n}\left(\hat{\beta}_{\text{HE}} - \beta\right) \to_d N(0, \frac{\mathbb{E}\left[u_i^2\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)\right]}{\left(\text{Var}\left(\theta_i\right)\right)^2}).$$

Taken together, the results in Theorem 3.3 and Proposition 3.4 establish the asymptotic properties of the HE estimator for the downstream regression. Theorem 3.3 shows that its asymptotic distribution is centered at zero and achieves the efficient variance, thus resolving the asymptotic bias from the EIV problem seen in the FE baseline. Proposition 3.4 further demonstrates that this property is robust, as the asymptotic distribution remains centered at zero even under the type of dependence that causes the HO estimator's distribution to be biased.

3.4 Inference of HE

This subsection further establishes the automatic inference property of $\hat{\boldsymbol{\theta}}_{\text{HE}}$. For implementation of inference in practice, we further require a consistent estimator for the asymptotic variance in Theorem 3.3. In this section, we show that $\hat{\boldsymbol{\theta}}_{\text{HE}}$ serves equivalently to the true $\boldsymbol{\theta}$ as the regressor in the conventional procedure of OLS regression.

Building on Assumption 3.1, additional regularity conditions are required for consistency of the variance estimator.

Assumption 3.2.

1. *J_i is independent of θ_i, u_i and $\epsilon_{i,j}$. $J_i \geq 3, a.s.$*
2. *$\mathbb{E}\left(u_i\right) = 0$, $\mathbb{E}(u_i\theta_i) = 0$. $\mathbb{E}\left[\theta_i^{k_1}u_i^{k_2}\right]$ is finite for any $1 \leq k_1 \leq 8$, $1 \leq k_2 \leq 4$.*
3. *$\mathbb{E}\left[\epsilon_{i,j} \mid \sigma_i^2\right] = 0$, $\mathbb{E}\left[\epsilon_{i,j}^2 \mid \sigma_i^2\right] = \sigma_i^2$, $\mathbb{E}\left[\left|\epsilon_{i,j}\right|^L \mid \sigma_i^2\right] \leq K\sigma_i^L$, for some $L \geq 3$. Also, $\epsilon_{i,j} \perp\!\!\!\perp \theta_i$.*
4. *$u_i \perp\!\!\!\perp \epsilon_{i,j} \big| \theta_i$.*
5. *θ_i has finite eighth moments.*
6. *σ_i^2 has finite eighth moments.*

Assumption 3.2.1, Assumption 3.2.3, Assumption 3.2.4, and Assumption 3.2.6 are the same as those in Assumption 3.1. We strengthen the moment conditions in Assumption 3.2.2 and Assumption 3.2.5 to ensure that the law of large numbers applies to higher order terms in the variance estimator.

Theorem 3.5. *Suppose the asymptotic framework satisfies* (3) *and* (4)*. Then under Assumption 3.2, the OLS standard error from regressing Y_i on $\hat{\theta}_{i,\mathrm{HE}}$ is consistent:*

$$\hat{\Omega} := \frac{\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,\mathrm{HE}} - \bar{\hat{\theta}}_{\mathrm{HE}}\right)^2 \hat{u}_i^2}{\left(\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,\mathrm{HE}} - \bar{\hat{\theta}}_{\mathrm{HE}}\right)^2\right)^2} \to_p \frac{\mathbb{E}\left[u_i^2\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)^2\right]}{\left(\mathrm{Var}\left(\theta_i\right)\right)^2},$$

where $\hat{u}_i$ is the OLS residual of regressing Y_i on $\hat{\theta}_{i,\mathrm{HE}}$:

$$\hat{u}_i := Y_i - \hat{\alpha} - \hat{\beta}\hat{\theta}_{i,\mathrm{HE}}.$$

Remark 3.1. Theorem 3.5 is proved in the same way as Theorem 3.3, by showing the properties of the estimators $\hat{\beta}_{c,\mathrm{HE}}$, and then draw the conclusion from consistency of $\hat{V}$ to $\mathrm{Var}\left(\theta_i\right)$. Here, the arguments are simpler than those for Theorem 3.3, because we only require convergence at a slower rate $o(1)$.

Notice that $\hat{\Omega}$ in Theorem 3.5 is precisely the Eicker-Huber-White variance estimator. If we regress Y_i on $\hat{\theta}_{i,\mathrm{HE}}$, the reported standard error is $\sqrt{\hat{\Omega}/n}$. Thus we can construct asymptotically valid confidence intervals

$$\hat{\beta}_{\mathrm{HE}} \pm z_{1-\alpha/2}\sqrt{\hat{\Omega}/n},$$

where $z_{1-\alpha/2}$ denotes the $1-\alpha/2$ quantile of the standard normal distribution. Theorem 3.5 ensures that the confidence interval has asymptotically nominal coverage. Specifically:

$$\Pr\left(\beta \in [\hat{\beta}_{\mathrm{HE}} - z_{1-\alpha/2}\sqrt{\hat{\Omega}/n}, \hat{\beta}_{\mathrm{HE}} + z_{1-\alpha/2}\sqrt{\hat{\Omega}/n}]\right) \to 1-\alpha.$$

In practice, inference with HE is straightforward. One treats $\hat{\boldsymbol{\theta}}_{\mathrm{HE}}$ as the latent $\boldsymbol{\theta}$ and regresses Y_i on $\hat{\theta}_{i,\mathrm{HE}}$ to obtain the OLS estimator $\hat{\beta}_{\mathrm{HE}}$ and its standard error. The resulting confidence interval for β is asymptotically valid.

While the focus of this paper is on downstream inference, it is worth noting that the HE estimator has favorable properties for estimating $\boldsymbol{\theta}$ itself. As an estimator motivated by empirical Bayes principles, its construction improves estimation accuracy. By correctly modeling all sources of heterogeneity, $\hat{\boldsymbol{\theta}}_{\text{HE}}$ is known to achieve a lower mean squared error (MSE) than the unshrunk FE, the HO, and the common-weight estimators, a standard result in that literature (see, e.g., Efron and Morris, 1973).

3.5 Common-Weight Shrinkage (CW)

As a final point of comparison, we analyse the common-weight (CW) shrinkage estimator, which serves as a benchmark. This estimator, motivated by the James-Stein estimator, shrinks each unit mean $\bar{X}_i$ towards the grand mean $\bar{X}$ by a common factor w:

$$\hat{\theta}_{i,\text{CW}} := w\bar{X}_i + (1-w)\bar{X},$$

where $\bar{X}$ denotes the grand mean of the sample. The weight w can be data-dependent as in the James-Stein estimator. For instance, common-weight shrinkage is applied in Chetty et al. (2014a,b), where the weight w is chosen from the best linear predictor of θ_i given $\bar{X}_i$. Inference of β is then performed by regressing Y_i on $\hat{\theta}_{i,\text{CW}}$.

We now give two examples of common weight w for shrinkage.

(i) Bias Correction Shrinkage

Regressing Y_i on $\hat{\theta}_{i,\text{CW}}$ leads to an OLS estimator of β given by

$$\hat{\beta}_{\text{CW}} := \frac{\widehat{\text{Cov}}\left(Y_i, \hat{\theta}_{i,\text{CW}}\right)}{\widehat{\text{Var}}\left(\hat{\theta}_{i,\text{CW}}\right)} = w^{-1} \underbrace{\frac{\widehat{\text{Cov}}\left(Y_i, \bar{X}_i\right)}{\widehat{\text{Var}}\left(\bar{X}_i\right)}}_{\hat{\beta}_{\text{FE}}}.$$

In effect, regressing Y_i on $\hat{\theta}_{i,\text{CW}}$ adjusts $\hat{\beta}_{\text{FE}}$ of regressing Y_i on $\bar{X}_i$ by a factor of w^{-1}. By a proper choice of shrinkage weight $w \approx \widehat{\text{Var}}\left(\theta_i\right) / \widehat{\text{Var}}\left(\bar{X}_i\right)$, the adjustment overlaps with the bias correction method for the EIV problem. Heuristically, in that case

$$\hat{\beta}_{\text{CW}} = \frac{\widehat{\text{Var}}\left(\bar{X}_i\right)}{\widehat{\text{Var}}\left(\theta_i\right)} \cdot \frac{\widehat{\text{Cov}}\left(Y_i, \bar{X}_i\right)}{\widehat{\text{Var}}\left(\bar{X}_i\right)} = \frac{\widehat{\text{Cov}}\left(Y_i, \bar{X}_i\right)}{\widehat{\text{Var}}\left(\theta_i\right)} \approx \frac{\widehat{\text{Cov}}\left(Y_i, \theta_i\right)}{\widehat{\text{Var}}\left(\theta_i\right)}.$$

The bias correction shrinkage method with the above w thus has the equivalent property as classical bias correction in EIV problems for downstream regressions.

(ii) IV Shrinkage

Another choice of w would also result in a shrinkage option that, when used as the regressor, refines the IV estimator for β. To see this, if we split the measurements for each unit i into two subsets, and treat the subset averages $\bar{X}_{i,1}$, $\bar{X}_{i,2}$ as the instrument and endogenous variable respectively, then we can construct an IV estimator:

$$\hat{\beta}_{\text{IV}} = \left(\frac{\widehat{\text{Cov}}\left(\bar{X}_{i,1}, \bar{X}_{i,2}\right)}{\widehat{\text{Var}}\left(\bar{X}_{i,1}\right)} \right)^{-1} \frac{\widehat{\text{Cov}}\left(Y_i, \bar{X}_{i,1}\right)}{\widehat{\text{Var}}\left(\bar{X}_{i,1}\right)} = \frac{\widehat{\text{Cov}}\left(Y_i, \bar{X}_{i,1}\right)}{\widehat{\text{Cov}}\left(\bar{X}_{i,1}, \bar{X}_{i,2}\right)} \approx \frac{\widehat{\text{Cov}}\left(Y_i, \theta_i\right)}{\widehat{\text{Var}}\left(\theta_i\right)}.$$

The IV estimator is adjusting the estimator of regressing Y_i on $\bar{X}_{i,1}$ by a factor. With a proper adjustment factor, we can exploit more information from regressing Y_i on $\bar{X}_i$ and achieve better estimation than using the subset $\bar{X}_{i,1}$.

Because of its connection to the bias correction method and the IV estimator, common-weight shrinkage can achieve as good performance as those methods for β, requiring as weak assumptions as them. Even though it ignores the heterogeneous signal-to-noise ratio and pool every individual at a uniform ratio, common-weight shrinkage provides valid inference on β. The following result formalizes this point. It is proved in Appendix B.

Proposition 3.6. *Suppose the asymptotic framework satisfies* (3) *and* (4)*. Under Assumption 3.1, we have*

$$\sqrt{n}\left(\hat{\beta}_{\text{CW}} - \beta\right) \to_d N\left(0, \frac{\mathbb{E}\left[u_i^2\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)^2\right]}{\left(\text{Var}\left(\theta_i\right)\right)^2}\right).$$

Therefore, the CW estimator also addresses the inferential problem from the FE baseline. Its primary distinction from the HE estimator is how it achieves this: the CW approach applies a uniform correction factor (equivalent to bias-correction or IV), while the HE estimator models unit-level heterogeneity. Despite these different mechanisms, both estimators are first-order asymptotically equivalent and achieve the same efficient asymptotic distribution.

4 Simulations

4.1 Simulation Design

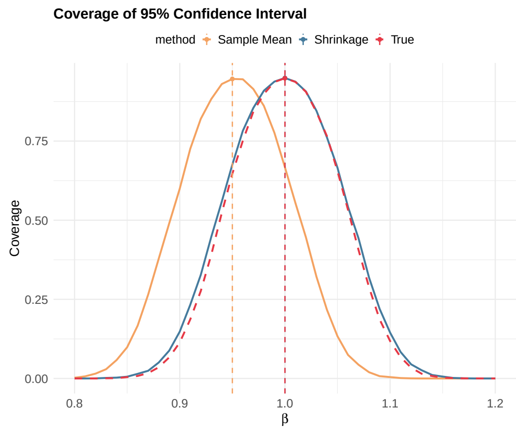
In the first set of simulations, we focus on the comparison of regressing on the shrinkage estimator $\hat{\theta}_{i,\mathrm{HE}}$, the sample mean $\bar{X}_i$, i.e. $\hat{\theta}_{i,\mathrm{FE}}$, and the true latent θ_i. In each of the $S = 3000$ simulations, we generate the data, run the regression of Y_i on generated regressors, and then report the 95% confidence intervals of β. Finally, for each value of β in the grid, we compute the coverage rate across all simulations, i.e. the proportion of simulations in which the value of β falls within the 95% confidence interval.

The number of measurements is fixed at $J_i = J = 20$ and the sample size is $n = 1000$. Here the ratio $\hat{\kappa} \approx 1.58$. We set the true θ_i drawn from the standard normal distribution, and the variance σ_i^2 drawn from $\chi^2(1)$. In the regression, we set $\alpha = 0$, $\beta = 1$, and draw u_i from a standard normal—thus homoskedastic—distribution.

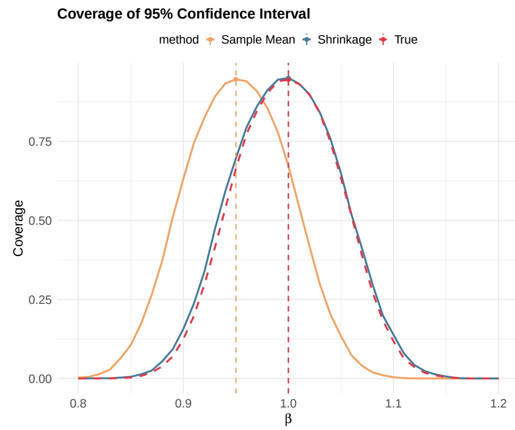
In the first setting, we generate the measurement error $\epsilon_{i,j}$ from a normal distribution, with all conditions in Assumption 3.1 satisfied. In the second setting, we set the measurement error $\epsilon_{i,j}$ still having the variance σ_i^2, but following centered linear transform $\sigma_i \frac{\chi^2(2)-2}{2}$. Note that the ratio $\epsilon_{i,j}/\sigma_i$ satisfies the moment conditions in Assumption 3.1. As $\epsilon_{i,j}$ follows a shifted Gamma distribution, which is non-normal and asymmetric, the purpose is to show that general distributions of measurement error are allowed as long as the moment conditions are satisfied.

In the second set of simulations, we compare bias and coverage rates across different methods, especially on the comparison of the individual-weight method HE and CW. We report the squared root scaled MSE of $\hat{\beta}$, coverage rate, bias, and MSE of $\hat{\boldsymbol{\theta}}$. The design we consider lets the heteroskedastic σ_i^2 be a uniform distribution from $\{1, 10\}$, with other settings the same as the previous case.

In the third set of simulations, we again compute the coverage rate for each method. We check the performance with random J_i drawn from Poisson(20), and with the sample size $n = 1000$. Here $\hat{\kappa}$ is approximately 1.58 but larger due to convexity. With other parameters unchanged, now we can have correlations between J_i and σ_i^2. In the first setting, we generate J_i and σ_i^2 independently, ensuring that all conditions in Assumption 3.1 are satisfied. In the second setting, we introduce a positive correlation between J_i and σ_i^2, reflecting possible endogenous choice of more measurements for lower precision, keeping conditions in Assumption D.1 satisfied.



(a) Normal Noise



(b) Gamma Noise

Note: Each curve represents the proportion of simulations in which the value of β on the x-axis falls within the 95% confidence interval. The dashed red line represents the infeasible case of regressing on the true θ_i. The blue line represents the proposed method of regressing on the shrinkage estimator $\hat{\theta}_i$ (HE). The orange line represents regressing on the sample mean $\bar{X}_i$ (FE).

Figure 1: Coverage Rates Under Different Noise Distributions

4.2 Results

The first set of coverage results are presented in Figure 1. Coverage for regressing on HE (the blue line) performs well in both normal and non-normal settings. The coverage rates are close to 95% at the true β, and the curve is always very close to the infeasible case (the dashed red line) of regressing on the true θ_i. In both cases, regressing on the sample mean $\bar{X}_i$ suffers from the attenuation bias, reflected by a shift to the left in the coverage curve (the orange line), though the spread is approximately correct, which aligns with Proposition 3.1.

The second set of results are reported in Table 1. The results indicate that HE has a smaller MSE of β, higher coverage rate, smaller bias, and smaller MSE of $\boldsymbol{\theta}$ than CW when the ratio $\sqrt{n}\mathbb{E}\left[J_i^{-1}\right]$ is reasonable. When n and J tend to be large, the performance of CW and HE are close, but HE always dominates and is even more preferable when the sample size is relatively small. As discussed in Section 3.5, we know that $\hat{\beta}_{\text{CW}}$ is first-order asymptotically equivalent to $\hat{\beta}_{\text{HE}}$. Combined with the simulation results, we can see that HE can achieve better performance in finite samples.

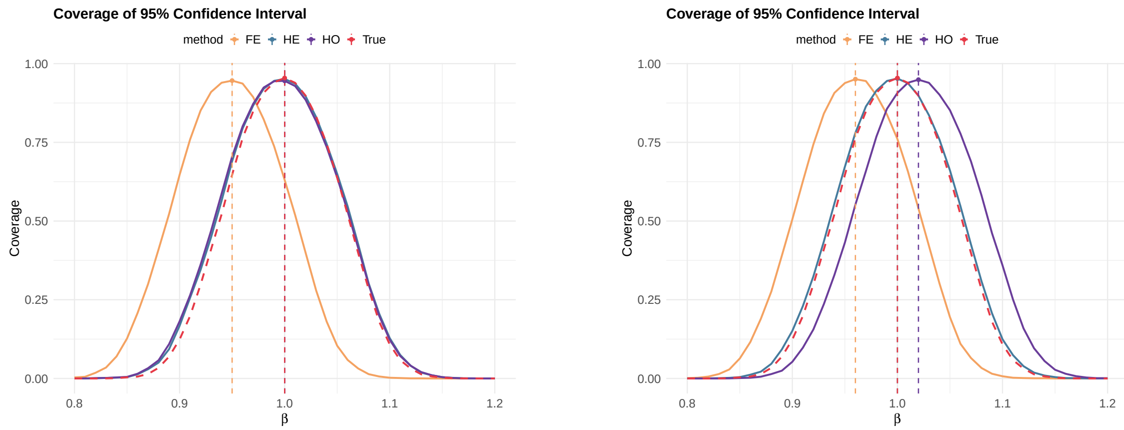
Table 1: Method Comparison Across Scenarios

Scenario	Method	$\sqrt{n \times \text{MSE}(\beta)}$	Coverage Rate (%)	Bias	MSE($\boldsymbol{\theta}$)
n=50, J=10	True θ_i	1.030	94.73	0.116	0
$\sqrt{n}/J \approx 0.71$	HE	1.497	91.90	0.166	0.322
	CW	2.335	87.83	0.225	0.374
	FE	2.690	27.67	0.354	0.552
n=225, J=10	True θ_i	1.016	94.17	0.054	0
$\sqrt{n}/J \approx 1.5$	HE	1.550	90.27	0.083	0.313
	CW	1.856	86.13	0.099	0.359
	FE	5.411	0.03	0.354	0.550
n=1000, J=20	True θ_i	1.009	95.13	0.025	0
$\sqrt{n}/J \approx 1.58$	HE	1.298	93.13	0.033	0.195
	CW	1.392	91.33	0.035	0.216
	FE	6.930	0.00	0.217	0.275

Note: This table compares the performance of different estimation methods across various scenarios, focusing on mean squared error (MSE), coverage rate, and bias. "True θ_i" represents the benchmark case where the true values of θ_i are known. HE denotes the heteroskedastic estimator, while CW refers to the common-weight estimator. FE represents the fixed-effects estimator. The square root of scaled MSE of β is computed as $\sqrt{n \times \text{MSE}(\beta)}$. Coverage rates are reported as percentages, and bias refers to the absolute deviation of the estimated β from its true value. The MSE of $\boldsymbol{\theta}$ measures the estimation error for individual effects. A larger $\sqrt{n}/J$ indicates a higher ratio of measurement error to sampling error.

The coverage results for random J_i are presented in Figure 2. We mainly focus on the curves for the proposed method (the blue line) and HO (the purple line). We can see that the proposed method works well in both settings, close to the infeasible case (the dashed red line). Instead, HO only works well in the first setting. When J_i and σ_i^2 are correlated, $\hat{\beta}_{\text{HO}}$ is biased, reflected by a shift to the right in the coverage plot for $\hat{\beta}_{\text{HO}}$. Similar as before, regressing on the sample mean $\bar{X}_i$ results in a leftward biased coverage curve. As discussed in Section 3.2, the asymptotic bias in HO is distinct from classical EIV attenuation bias, as it shifts the estimate in the opposite direction and causes amplification.

Overall, the simulation results confirm the theoretical findings in Section 3 and



(a) $J_i \perp\!\!\!\perp \sigma_i^2$

(b) J_i correlated with σ_i^2

Note: Each curve represents the proportion of simulations in which the value of β on the x-axis falls within the 95% confidence interval. The dashed red line represents the infeasible case of regressing on the true θ_i. The blue line represents the method of regressing on the heteroskedastic individual-weight shrinkage estimator $\hat{\theta}_{i,\text{HE}}$ (HE). The purple line represents the regressing on the homoskedastic individual-weight shrinkage estimator $\hat{\theta}_{i,\text{HO}}$ (HO). The orange line represents regressing on the sample mean $\bar{X}_i$ (FE).

Figure 2: Coverage Rates Under Different Dependence

support the extension of Assumption 3.2 regarding the normality of $\epsilon_{i,j}$. Additionally, they validate the theoretical framework in Section 3 concerning the dependence of J_i and σ_i^2. The proposed method works well in both normal and non-normal settings, and is robust to the correlation between J_i and σ_i^2. It outperforms FE, CW and HO, and is close to the infeasible best case.

5 Empirical Application: Firm Discrimination

In this section, we use the HE method in the context of Kline et al. (2022) about the extent to which large U.S. employers systemically discriminate job applicants based on race. Their study utilizes correspondence audits, where fictitious resumes with randomized racial identifiers are sent to employers to measure differences in callback rates. The racial contact gap, defined as the difference in callback probabilities between racial groups, serves as the primary latent variable, analogous to value-added in our setting.

5.1 Data

We use the panel dataset in Kline et al. (2022) on an experiment that sends fictitious applications to jobs posted by 108 of the largest U.S. employers. For each firm in each wave, about 25 entry-level vacancies were sampled and, for each vacancy, 8 job applications with randomly assigned characteristics were sent to the employer. Sampling was organized in 5 waves. Focusing on firms sampled in all waves yields a balanced panel of $n = 70$ firms over 5 waves. Applications were sent in pairs, one randomly assigned a distinctively White name and the other a distinctively Black name. The primary outcome is whether the employer attempted to contact the applicant within 30 days of applying. The racial contact gap is defined as the firm-level difference between the contact rate (the ratio of number of contacts and number of received applications) for White and that for Black applications. We follow the model similar to Kline et al. (2022), where the racial contact gap is given by

$$\bar{X}_i = \theta_i + \bar{\epsilon}_i, \quad \bar{\epsilon}_i \sim N\left(0, \frac{\sigma_i^2}{J_i}\right),$$

with normality arising from the central limit theorem approximation.

5.2 Estimation

In our analysis, we estimate the predictive effect of callback probability in wave t, for $t = 1, 2, 3$, on callback probability in wave $t + 2$. Given the model setup, we expect the regression coefficient to be close to 1. For each wave t, we first apply shrinkage estimation, and subsequently regress on these estimates. We compare the performance of our method with the fixed effects (FE) estimator across the three waves. Since the discrimination gap is computed from pairs of job applications, we have $J = 100$, resulting in a small ratio of measurement error to sampling error, $\sqrt{n}/J = 0.08$.

The results presented in Table 2 indicate that regression on the shrinkage estimates ($\hat{\theta}_{i,\text{HE}}$) yields coefficients closer to 1, along with lower mean squared error (MSE). Despite a small $\sqrt{n}/J$, FE still exhibits attenuation bias. In terms of inference, the proposed method robustly rejects the null hypothesis of no predictive effect, whereas FE fails to reject this null hypothesis for waves 1 and 2. These findings demonstrate that the proposed estimator improves both the accuracy of estimation and the validity of inference for the regression coefficient.

Table 2: Regression Results for Different Waves Predicting $t+2$

	$\bar{X}_i$				$\hat{\theta}_{i,\text{HE}}$			
Wave	β	SE	CI	p-value	β	SE	CI	p-value
Wave 1	0.150	0.100	[-0.047, 0.347]	0.140	0.987	0.385	[0.232, 1.742]	0.013
Wave 2	0.092	0.110	[-0.124, 0.308]	0.406	0.883	0.363	[0.171, 1.594]	0.017
Wave 3	0.415	0.125	[0.170, 0.659]	0.001	2.186	0.963	[0.299, 4.073]	0.026
MSE(β)	0.630				0.474			

Note: This table presents regression results for firm discrimination in wave t predicting firm discrimination in wave $t+2$. Columns labeled $\bar{X}_i$ represent regressions using the raw sample mean, while columns labeled $\hat{\theta}_{i,\text{HE}}$ correspond to regressions using the shrinkage estimates (HE). The coefficients (β) indicate the estimated effect of firm discrimination in wave t on wave $t+2$. The standard error (SE), 95% confidence interval (CI), and p-value are also reported, based on the Eicker–Huber–White variance estimator.

6 Application: School Value-Added in Pakistan

In this section, we use the HE method in the context of Andrabi et al. (2025) about estimating the school value-added in Pakistan.

6.1 Data and Empirical Setting

The analysis utilizes the LEAPS project dataset, a rich longitudinal panel from rural Punjab, Pakistan. The data track 71,677 child-year test scores across more than 800 schools from 2003–2006, forming one of the largest such panels in a developing country. This setting is characterized by the rapid emergence of a private school market, making the estimation of school quality (value-added) and its perceived return (school fees) a central question of interest. We use the school-year level sample from their analysis, which consists of $n = 1158$ observations. We also have $\mathbb{E}_n\left[J_i^{-1}\right] = 0.120$. Therefore $\hat{\kappa} = \sqrt{n}\mathbb{E}_n\left[J_i^{-1}\right] \approx 4.08$.

6.2 Estimation and Results

We replicate and extend the primary downstream analysis in Andrabi et al. (2025), which is a regression of private school fees on estimated SVA. The latent variable θ_i represents the true SVA of a school, and the downstream regression investigates whether schools with higher SVA command higher fees.

We compare two individualized shrinkage estimators. The first is HO, as used in the original study. The second is HE, which is fully individualized by allowing for heterogeneity in both J_i and the noise variance σ_i^2.

The results are presented in Table 3 (full sample) and Table 4 (a selected sample restricting $20 \leq J_i \leq 80$, following the original paper). The tables compare the HE estimator (left panel) and the HO estimator (right panel) in both bivariate specifications and specifications including household-level controls (parental education and asset index).

Across all specifications, our results confirm the central economic finding of Andrabi et al. (2025): SVA is a large, positive, and statistically significant predictor of private school fees. For example, in the full-sample specification with controls (Table 3, Column 2), the HE point estimate is 793.013 and is statistically significant at the 1% level.

7 Conclusion

This paper investigates the inferential properties of individualized shrinkage estimators when used as downstream regressors, a common but not fully understood empirical practice. We formalize the conditions under which this plug-in approach yields valid downstream inference. Our central finding is that a correctly specified and fully individualized shrinkage estimator yields an asymptotically efficient estimator of the downstream coefficient. Crucially, we show that conventional OLS standard inference are asymptotically valid, justifying standard empirical practice. This result stands in contrast to the biased inference from the unshrunk FE baseline and the simpler individual-weight (HO) estimators that can fail when noise variance is correlated with the number of measurements. We apply our method to data on firm discrimination and school value-added and show that it improves the estimation and inference.

Our analysis provides a formal bridge between the empirical Bayes literature,

Table 3: Regression Results of School Value-Added on Private School Fees

	(1)	(2)
	Dep. Var.: School Fees	
Empirical Bayes SVA	1054.128***	793.013***
	(298.150)	(264.052)
Mean Mother Education		2.314
		(123.829)
Mean Father Education		203.270
		(142.282)
Mean HH Asset Index		263.334***
		(46.927)
Adjusted R^2	0.23	0.29
Number of Observations	1154	1144
Number of Clusters	315	315

	(1)	(2)
	Dep. Var.: School Fees	
Empirical Bayes SVA	991.371***	719.858**
	(320.963)	(280.882)
Mean Mother Education		4.697
		(122.849)
Mean Father Education		218.651
		(142.686)
Mean HH Asset Index		263.074***
		(46.820)
Adjusted R^2	0.22	0.29
Number of Observations	1158	1148
Number of Clusters	318	318

Notes: Each column reports coefficients from regressions of private school fees on estimated school value-added. The left panel uses the HE estimator, and the right panel uses the HO estimator following Andrabi et al. (2025). Columns (2) add controls for mean mother education, mean father education, and mean household asset index. Standard errors (in parentheses) are clustered at the village level. *, **, and *** denote significance at the 10%, 5%, and 1% levels, respectively.

where shrinkage estimators were developed to improve estimation accuracy, and the common empirical practice of using these estimates for downstream inference. The key takeaway for practitioners is that while the plug-in approach can be valid and efficient, its robustness depends critically on the specification of the shrinkage estimator. For applied researchers, our results provide a clear theoretical foundation and a practical guide for obtaining valid inference when using shrinkage estimates as regressors in linear models. Extending this method to nonlinear settings is a natural yet nontrivial direction, which we are studying in ongoing work.

Table 4: Regression Results of School Value-Added on Private School Fees (Selected Sample)

	(1)	(2)
	Dep. Var.: School Fees	
Empirical Bayes SVA	1622.126***	1358.509***
	(454.324)	(391.156)
Mean Mother Education		-118.275
		(230.302)
Mean Father Education		393.320
		(271.010)
Mean HH Asset Index		359.615***
		(83.513)
Adjusted R^2	0.31	0.39
Number of Observations	592	591
Number of Clusters	150	150

	(1)	(2)
	Dep. Var.: School Fees	
Empirical Bayes SVA	1569.700***	1320.180***
	(451.274)	(384.742)
Mean Mother Education		-119.644
		(230.177)
Mean Father Education		402.032
		(270.856)
Mean HH Asset Index		360.131***
		(83.527)
Adjusted R^2	0.31	0.39
Number of Observations	592	591
Number of Clusters	150	150

Notes: The sample is restricted to schools with between 20 and 80 students, following Andrabi et al. (2025). The left panel reports the HE estimator, and the right panel replicates the HO estimates from the original study. Standard errors (in parentheses) are clustered at the village level. *, **, and *** denote significance at the 10%, 5%, and 1% levels, respectively.

A Proofs of Main Results

A.1 Proofs for FE

Proof of Proposition 3.1. The regularity assumptions are Assumption 3.1. Firstly, for simplicity we abbreviate notations and denote $\hat{\beta}_{\text{FE}}$ as $\hat{\beta}$, and $\text{Var}\left(\theta_i\right)$ as V.

$$
\begin{aligned}
\sqrt{n}\left(\hat{\beta}-\beta\right) &= \left(\frac{1}{n}\sum_{i=1}^{n}\left(\bar{X}_i-\bar{X}\right)^2\right)^{-1} \\
&\quad \left(\beta\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\bar{X}_i-\bar{X}\right)\left(\theta_i-\bar{\theta}\right)-\beta\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\bar{X}_i-\bar{X}\right)^2+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\bar{X}_i-\bar{X}\right)\left(u_i-\bar{u}\right)\right) \\
&= \frac{\beta\sqrt{n}T_{1,n}-\beta\sqrt{n}T_{2.n}+\sqrt{n}T_{3,n}}{T_{2,n}}.
\end{aligned}
$$

Firstly, from Lemma B.1 and Lemma B.3, we have for the denominator,

$$\begin{aligned} T_{2,n} &= \hat{V} + \frac{n-1}{n^2}\sum_{i=1}^{n}\frac{1}{J_i}\hat{\sigma}_i^2 \\ &= \hat{V} + O_p(n^{-1/2}) \\ &= V + O_p(n^{-1/2}). \end{aligned}$$

For the numerator terms, by properties from Lemma B.1,

$$\begin{aligned} \sqrt{n}T_{1,n} &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)^2 + \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\bar{\epsilon}_i\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right) \\ &\quad - \sqrt{n}\left(\bar{\theta} - \mathbb{E}\left[\theta_i\right]\right)^2 - \sqrt{n}\left(\bar{\theta} - \mathbb{E}\left[\theta_i\right]\right)\bar{\epsilon} \\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)^2 + o_p(1), \end{aligned}$$

$$\begin{aligned} \sqrt{n}T_{3,n} &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)u_i + \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\bar{\epsilon}_i u_i \\ &\quad - \sqrt{n}\left(\bar{\theta} - \mathbb{E}\left[\theta_i\right]\right)\bar{u} - \sqrt{n}\bar{\epsilon}\bar{u} \\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)u_i + o_p(1). \end{aligned}$$

Combined with the proof of Lemma B.3, we have

$$\begin{aligned} \sqrt{n}T_{2,n} &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)^2 + \bar{\epsilon}_i^2\right] - \sqrt{n}\left(\bar{\theta} - \mathbb{E}\left[\theta_i\right]\right)^2 - \sqrt{n}\bar{\epsilon}^2 + \frac{2}{\sqrt{n}}\sum_{i=1}^{n}\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)\bar{\epsilon}_i \\ &\quad - 2\sqrt{n}\left(\bar{\theta} - \mathbb{E}\left[\theta_i\right]\right)\bar{\epsilon} \\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)^2 + \bar{\epsilon}_i^2\right] + o_p(1). \end{aligned}$$

Therefore, the numerator is

$$\begin{aligned}
&\beta\sqrt{n}T_{1,n} - \beta\sqrt{n}T_{2,n} + \sqrt{n}T_{3,n} \\
&= -\beta\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\bar{\epsilon}_i^2 + \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)u_i + o_p(1) \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)u_i + \beta\frac{1}{J_i}\sigma_i^2 - \beta\bar{\epsilon}_i^2\right] - \beta\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{J_i}\sigma_i^2 + o_p(1) \\
&:= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi_i - \beta\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{J_i}\sigma_i^2 + o_p(1).
\end{aligned}$$

Here for the second term, by Chebyshev's inequality, for any $s > 0$ we have

$$\Pr\left(\sqrt{n}\left|\frac{1}{n}\sum_{i=1}^{n}\frac{1}{J_i}\sigma_i^2 - \mathbb{E}\left[\frac{1}{J_i}\sigma_i^2\right]\right| > s\right) \leq \frac{\mathbb{E}\left[\frac{1}{J_i^2}\right]\mathbb{E}\left[\sigma_i^4\right]}{s^2} \to 0,$$

and also,

$$\sqrt{n}\mathbb{E}\left[\frac{1}{J_i}\sigma_i^2\right] = \sqrt{n}\mathbb{E}\left[\frac{1}{J_i}\right]\mathbb{E}\left[\sigma_i^2\right] \to \kappa\mathbb{E}\left[\sigma_i^2\right].$$

Therefore, the second term is $-\kappa\beta\mathbb{E}\left[\sigma_i^2\right] + o_p(1)$.

For ξ_i, since

$$\mathbb{E}\left[\xi_i\right] = 0,$$

$$\begin{aligned}
\mathbb{E}\left[\xi_i^2\right] &= \mathbb{E}\left[u_i^2\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)^2\right] + \beta^2\mathbb{E}\left[\left(\frac{1}{J_i}\sigma_i^2\right)^2\right] + \beta^2\mathbb{E}\left[\bar{\epsilon}_i^4\right] - 2\beta^2\mathbb{E}\left[\frac{1}{J_i}\sigma_i^2\bar{\epsilon}_i^2\right] \\
&= \mathbb{E}\left[u_i^2\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)^2\right] + o(1) - 2\beta^2\mathbb{E}\left[\frac{1}{J_i^2}\sigma_i^4\right] \\
&= \mathbb{E}\left[u_i^2\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)^2\right] + o(1),
\end{aligned} \tag{6}$$

where the last line follows from Lemma B.1.

Therefore, for $\left(n\mathbb{E}\left[\xi_i^2\right]\right)^{-1/2}\sum_{i=1}^{n}\xi_i$ in the triangular array, by the Lindeberg-Feller theorem (See Ferguson (2017) p.27), because the Lindberg condition holds below

$$\frac{1}{n\mathbb{E}\left[\xi_i^2\right]}\sum_{i=1}^{n}\mathbb{E}\left\{\xi_i^2\mathbb{1}\left(|\xi_i|>s\sqrt{n\mathbb{E}\left[\xi_i^2\right]}\right)\right\}=\frac{1}{\mathbb{E}\left[\xi_i^2\right]}\mathbb{E}\left\{\xi_i^2\mathbb{1}\left(|\xi_i|>s\sqrt{n\mathbb{E}\left[\xi_i^2\right]}\right)\right\}$$
$$\to 0,\ \forall s>0,$$

which is derived from (6) and the dominated convergence theorem, then we have

$$\frac{1}{\sqrt{n\mathbb{E}\left[\xi_i^2\right]}}\sum_{i=1}^{n}\xi_i \to_d N(0,1).$$

By (6) and Slutsky's theorem,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi_i-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{J_i}\sigma_i^2 \to_d N(-\kappa\beta\mathbb{E}\left[\sigma_i^2\right],\mathbb{E}\left[u_i^2\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^2\right]).$$

Then combined with the denominator, by Slutsky's theorem,

$$\sqrt{n}\left(\hat{\beta}-\beta\right)=\frac{\beta\sqrt{n}T_{1,n}-\beta\sqrt{n}T_{2,n}+\sqrt{n}T_{3,n}}{T_{2,n}}\to_d N(-\kappa\beta\frac{\mathbb{E}\left[\sigma_i^2\right]}{V},\frac{\mathbb{E}\left[u_i^2\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^2\right]}{V^2}).$$

□

A.2 Proofs for Asymptotic Normality and Inference

Proof of Lemma 3.1. Firstly, for simplicity we abbreviate notations and denote $\hat{\beta}_{c,\mathrm{HE}}$ as $\hat{\beta}_c$, $\hat{\theta}_{i,c,\mathrm{HE}}$ as $\hat{\theta}_{i,c}$, and $\mathrm{Var}\left(\theta_i\right)$ as V.

We have

$$\begin{aligned}
&\sqrt{n}\left(\hat{\beta}_c-\beta\right)\\
=&\left(\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c}-\bar{\hat{\theta}}_c\right)^2\right)^{-1}\\
&\left(\beta\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c}-\bar{\hat{\theta}}_c\right)\left(\theta_i-\bar{\theta}\right)-\beta\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c}-\bar{\hat{\theta}}_c\right)^2+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c}-\bar{\hat{\theta}}_c\right)\left(u_i-\bar{u}\right)\right)\\
:=&\frac{\beta\sqrt{n}T_{1,n}-\beta\sqrt{n}T_{2,n}+\sqrt{n}T_{3,n}}{T_{2,n}}.
\end{aligned}$$

Respectively, by properties from Lemma B.1,

$$
\begin{aligned}
\sqrt{n} T_{1,n} &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} c_i \left[\left(\theta_i - \bar{\theta}\right)^2 + \left(\bar{\epsilon}_i - \bar{\epsilon}\right) \left(\theta_i - \bar{\theta}\right) \right] \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} c_i \left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)^2 + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} c_i \bar{\epsilon}_i \left(\theta_i - \mathbb{E}\left[\theta_i\right]\right) \\
&\quad - 2\sqrt{n} \left(\bar{\theta} - \mathbb{E}\left[\theta_i\right]\right) \frac{1}{n} \sum_{i=1}^{n} c_i \left(\theta_i - \mathbb{E}\left[\theta_i\right]\right) + \sqrt{n} \left(\bar{\theta} - \mathbb{E}\left[\theta_i\right]\right)^2 \frac{1}{n} \sum_{i=1}^{n} c_i \\
&\quad - \sqrt{n} \left(\bar{\theta} - \mathbb{E}\left[\theta_i\right]\right) \frac{1}{n} \sum_{i=1}^{n} c_i \bar{\epsilon}_i - \sqrt{n} \bar{\epsilon} \frac{1}{n} \sum_{i=1}^{n} c_i \left(\theta_i - \mathbb{E}\left[\theta_i\right]\right) + \sqrt{n} \bar{\epsilon} \left(\bar{\theta} - \mathbb{E}\left[\theta_i\right]\right) \frac{1}{n} \sum_{i=1}^{n} c_i \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} c_i \left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)^2 + o_p(1),
\end{aligned}
$$

$$
\begin{aligned}
T_{2,n} &= \frac{1}{n}\sum_{i=1}^{n}\left(c_i\left(\theta_i-\bar{\theta}\right)+c_i\left(\bar{\epsilon}_i-\bar{\epsilon}\right)\right)^2+\left[\frac{1}{n}\sum_{i=1}^{n}\left(c_i\left(\theta_i-\bar{\theta}+\bar{\epsilon}_i-\bar{\epsilon}\right)\right)\right]^2 \\
&= \frac{1}{n}\sum_{i=1}^{n}c_i^2\left[\left(\theta_i-\bar{\theta}\right)^2+\left(\bar{\epsilon}_i-\bar{\epsilon}\right)^2+2\left(\theta_i-\bar{\theta}\right)\left(\bar{\epsilon}_i-\bar{\epsilon}\right)\right] \\
&\quad+\left[\frac{1}{n}\sum_{i=1}^{n}c_i\theta_i-\bar{\theta}\bar{c}+\frac{1}{n}\sum_{i=1}^{n}c_i\bar{\epsilon}_i-\bar{\epsilon}\bar{c}\right]^2 \\
&= \frac{1}{n}\sum_{i=1}^{n}c_i^2\left[\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^2+\bar{\epsilon}_i^2+2\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)\bar{\epsilon}_i\right] \\
&\quad-2\left(\bar{\theta}-\mathbb{E}\left[\theta_i\right]\right)\frac{1}{n}\sum_{i=1}^{n}c_i^2\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)+\left(\bar{\theta}-\mathbb{E}\left[\theta_i\right]\right)^2\frac{1}{n}\sum_{i=1}^{n}c_i^2 \\
&\quad-2\bar{\epsilon}\frac{1}{n}\sum_{i=1}^{n}c_i^2\bar{\epsilon}_i+\bar{\epsilon}^2\frac{1}{n}\sum_{i=1}^{n}c_i^2 \\
&\quad-2\left(\bar{\theta}-\mathbb{E}\left[\theta_i\right]\right)\frac{1}{n}\sum_{i=1}^{n}c_i^2\bar{\epsilon}_i-2\bar{\epsilon}\frac{1}{n}\sum_{i=1}^{n}c_i^2\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)+2\bar{\epsilon}\left(\bar{\theta}-\mathbb{E}\left[\theta_i\right]\right)\frac{1}{n}\sum_{i=1}^{n}c_i^2 \\
&\quad+\left[\mathbb{E}\left[\theta_i\right]+O_p(n^{-1/2})-\left(\mathbb{E}\left[\theta_i\right]+O_p(n^{-1/2})\right)\left(1+O_p(n^{-1/2})\right)+o_p(n^{-1/2})\right]^2 \\
&= \frac{1}{n}\sum_{i=1}^{n}c_i^2\left[\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^2+\bar{\epsilon}_i^2\right]+o_p(n^{-1/2}) \\
&\quad-2O_p(n^{-1/2})O_p(n^{-1/2})+O_p(n^{-1})\left[1+O_p(n^{-1/2})\right] \\
&\quad-2o_p(n^{-1/2})o_p(n^{-1/2})+o_p(n^{-1})\left[1+O_p(n^{-1/2})\right] \\
&\quad-2O_p(n^{-1/2})o_p(n^{-1/2})-2o_p(n^{-1/2})O_p(n^{-1/2})+2o_p(n^{-1/2})O_p(n^{-1/2})\left[1+O_p(n^{-1/2})\right] \\
&\quad+O_p(n^{-1}) \\
&= \frac{1}{n}\sum_{i=1}^{n}c_i^2\left[\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^2+\bar{\epsilon}_i^2\right]+o_p(n^{-1/2}).
\end{aligned}
$$

$$
\begin{aligned}
\sqrt{n}T_{3,n} = &\frac{1}{\sqrt{n}}\sum_{i=1}^{n} c_i\left[\left(\theta_i-\bar{\theta}\right)(u_i-\bar{u})+(\bar{\epsilon}_i-\bar{\epsilon})(u_i-\bar{u})\right]\\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} c_i\left[\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)u_i\right]+\frac{1}{\sqrt{n}}\sum_{i=1}^{n} c_i\bar{\epsilon}_i u_i\\
&-\sqrt{n}\left(\bar{\theta}-\mathbb{E}\left[\theta_i\right]\right)\frac{1}{n}\sum_{i=1}^{n} c_i u_i-\sqrt{n}\bar{u}\frac{1}{n}\sum_{i=1}^{n} c_i\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)+\sqrt{n}\bar{u}\left(\bar{\theta}-\mathbb{E}\left[\theta_i\right]\right)\frac{1}{n}\sum_{i=1}^{n} c_i\\
&-\sqrt{n}\bar{\epsilon}\frac{1}{n}\sum_{i=1}^{n} c_i u_i-\sqrt{n}\bar{u}\frac{1}{n}\sum_{i=1}^{n} c_i\bar{\epsilon}_i+\sqrt{n}\bar{u}\bar{\epsilon}\frac{1}{n}\sum_{i=1}^{n} c_i\\
= &\frac{1}{\sqrt{n}}\sum_{i=1}^{n} c_i\left[\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)u_i\right]+o_p(1).
\end{aligned}
$$

Therefore, for the denominator we have

$$
\begin{aligned}
T_{2,n} &= \frac{1}{n}\sum_{i=1}^{n} c_i^2\left[\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^2+\bar{\epsilon}_i^2\right]+o_p(n^{-1/2})\\
&= \frac{1}{n}\sum_{i=1}^{n} c_i^2\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^2+O_p(n^{-1/2})\\
&= V+o_p(1).
\end{aligned}
$$

Also, combined with Lemma B.2, the numerator is

$$
\begin{aligned}
&\beta\sqrt{n}T_{1,n}-\beta\sqrt{n}T_{2,n}+\sqrt{n}T_{3,n}\\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\beta c_i\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^2-\beta c_i^2\left[\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^2+\bar{\epsilon}_i^2\right]+c_i\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)u_i\right]+o_p(1)\\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\beta c_i\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^2-\beta c_i^2\left[\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^2+\frac{1}{J_i}\hat{\sigma}_i^2\right]+c_i\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)u_i\right]+o_p(1)\\
&:= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi_i+o_p(1).
\end{aligned}
$$

For ξ_i, since

$$\begin{aligned}
\mathbb{E}\left[\xi_i\right] &= \beta\mathbb{E}\left[c_i\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)^2\right] - \beta\mathbb{E}\left[c_i^2\left(\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)^2 + \frac{1}{J_i}\hat{\sigma}_i^2\right)\right] \\
&= \beta V\mathbb{E}\left(c_i\right) - \beta\mathbb{E}\left[c_i^2\left(V + \frac{1}{J_i}\hat{\sigma}_i^2\right)\right] \\
&= \beta V\mathbb{E}\left(c_i\right) - \beta\mathbb{E}\left[Vc_i\right] \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[\xi_i^2\right] &= \beta^2\mathbb{E}\left[\left(c_i - c_i^2\right)^2\right]\mathbb{E}\left[\left(\theta_i - \mathbb{E}\left(\theta_i\right)\right)^4\right] + \beta^2\mathbb{E}\left[\frac{1}{J_i^2}c_i^4\hat{\sigma}_i^4\right] \\
&\quad + \mathbb{E}\left[c_i^2\right]\mathbb{E}\left[u_i^2\left(\theta - \mathbb{E}\left[\theta_i\right]\right)^2\right] \\
&\quad - 2\beta^2 V\mathbb{E}\left[\frac{1}{J_i}\hat{\sigma}_i^2 c_i^2\left(c_i - c_i^2\right)\right] \\
&= o(1) + \beta^2\mathbb{E}\left[c_i^4\left(\frac{1}{J_i}\hat{\sigma}_i^2\right)^2\right] \\
&\quad + \mathbb{E}\left[c_i^2\right]\mathbb{E}\left[u_i^2\left(\theta - \mathbb{E}\left[\theta_i\right]\right)^2\right] - 2\beta^2 V\mathbb{E}\left[\frac{1}{J_i}\hat{\sigma}_i^2 c_i^2\left(c_i - c_i^2\right)\right] \\
&= \mathbb{E}\left[u_i^2\left(\theta - \mathbb{E}\left[\theta_i\right]\right)^2\right] + o(1), \qquad (7)
\end{aligned}$$

where the last line follows from Lemma B.1 and also $0 < c_i < 1$.

Therefore, for $\left(n\mathbb{E}\left[\xi_i^2\right]\right)^{-1/2}\sum_{i=1}^n \xi_i$ in the triangular array, by the Lindeberg-Feller theorem (See Ferguson (2017) p.27), because the Lindberg condition holds below

$$\begin{aligned}
\frac{1}{n\mathbb{E}\left[\xi_i^2\right]}\sum_{i=1}^n \mathbb{E}\left\{\xi_i^2\mathbb{1}\left(|\xi_i| > s\sqrt{n\mathbb{E}\left[\xi_i^2\right]}\right)\right\} &= \frac{1}{\mathbb{E}\left[\xi_i^2\right]}\mathbb{E}\left\{\xi_i^2\mathbb{1}\left(|\xi_i| > s\sqrt{n\mathbb{E}\left[\xi_i^2\right]}\right)\right\} \\
&\to 0,\ \forall s > 0,
\end{aligned}$$

which is derived from (7) and the dominated convergence theorem, then we have

$$\frac{1}{\sqrt{n\mathbb{E}\left[\xi_i^2\right]}}\sum_{i=1}^n \xi_i \to_d N(0,1).$$

By (7) and Slutsky's theorem,

$$\frac{1}{\sqrt{n\sigma_u^2 V}}\sum_{i=1}^{n}\xi_i \rightarrow_d N(0,1).$$

Therefore,

$$\beta\sqrt{n}T_{1,n} - \beta\sqrt{n}T_{2,n} + \sqrt{n}T_{3,n} \rightarrow_d N(0, \mathbb{E}\left[u_i^2\left(\theta - \mathbb{E}\left[\theta_i\right]\right)^2\right]).$$

Then combined with the denominator, by Slutsky's theorem,

$$\sqrt{n}\left(\hat{\beta}_c - \beta\right) = \frac{\beta\sqrt{n}T_{1,n} - \beta\sqrt{n}T_{2,n} + \sqrt{n}T_{3,n}}{T_{2,n}} \rightarrow_d N(0, \frac{\mathbb{E}\left[u_i^2\left(\theta - \mathbb{E}\left[\theta_i\right]\right)^2\right]}{V^2}).$$

□

Proof of Theorem 3.3. Firstly, for simplicity we abbreviate notations and denote $\hat{\beta}_{c,\text{HE}}$ as $\hat{\beta}_c$, $\hat{\beta}_{\text{HE}}$ as $\hat{\beta}$, $\hat{\theta}_{i,c,\text{HE}}$ as $\hat{\theta}_{i,c}$, $\hat{\theta}_{i,\text{HE}}$ as $\hat{\theta}_i$ and $\text{Var}\left(\theta_i\right)$ as V. We then prove the asymptotics by showing

$$\sqrt{n}\left(\hat{\beta} - \beta\right) \rightarrow_p \sqrt{n}\left(\hat{\beta}_c - \beta\right).$$

By taking the difference, it's equivalent to

$$\sqrt{n}\hat{\beta} - \sqrt{n}\hat{\beta}_c = o_p(1).$$

It suffices to show for the numerator part

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\hat{\theta}_i - \bar{\hat{\theta}}\right)\left(Y_i - \bar{Y}\right) \rightarrow_p \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c} - \bar{\hat{\theta}}_c\right)\left(Y_i - \bar{Y}\right),$$

and then show for the denominator part

$$\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_i - \bar{\hat{\theta}}\right)^2 \rightarrow_p \frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c} - \bar{\hat{\theta}}_c\right)^2.$$

First we show the numerator part. Since $\hat{\theta}_i$ and $\hat{\theta}_{i,c}$ can be viewed as function values given $t = \hat{V}$ and $t = V$ for the function of t:

$$\bar{X} + \frac{t}{\frac{1}{J_i}\hat{\sigma}_i^2 + t}\left(\bar{X}_i - \bar{X}\right),$$

the first order derivative of which is denoted as

$$A_i(t) \coloneqq \frac{\frac{1}{J_i}\hat{\sigma}_i^2}{\left(\frac{1}{J_i}\hat{\sigma}_i^2 + t\right)^2}\left(\bar{X}_i - \bar{X}\right),$$

for simplicity of notations, we denote the whole numerator part as a function $f_n(t)$, and what we want to show is

$$f_n(\hat{V}) - f_n(V) = o_p(1).$$

By the mean value expansion theorem, for $\tilde{V}$ such that $\left|\tilde{V} - V\right| \leq \left|\hat{V} - V\right|$,

$$\begin{aligned}\left|f_n\left(\hat{V}\right) - f_n\left(V\right)\right| &= \left|\hat{V} - V\right| \left|f_n'\left(\tilde{V}\right)\right| \\ &\leq \sqrt{n}\left|\hat{V} - V\right| \frac{1}{n}\sum_{i=1}^{n} \frac{\frac{1}{J_i}\hat{\sigma}_i^2}{\left(\frac{1}{J_i}\hat{\sigma}_i^2 + \tilde{V}\right)^2} \left|\left(\bar{X}_i - \bar{X}\right)\left(Y_i - \bar{Y}\right)\right|\end{aligned}$$

For any $s > 0$, there exists $\delta > 0$ such that $V - \delta > 0$, and we have

$$
\begin{aligned}
&\Pr\left(\left|f_n\left(\hat{V}\right) - f_n(V)\right| > s\right) \\
&\leq \Pr\left(\left|f_n\left(\hat{V}\right) - f_n(V)\right| > s, \left|\hat{V} - V\right| \leq \delta\right) + \Pr\left(\left|\hat{V} - V\right| > \delta\right) \\
&\leq \Pr\left(\sqrt{n}\left|\hat{V} - V\right| \frac{1}{n}\sum_{i=1}^{n} \frac{\frac{1}{J_i}\hat{\sigma}_i^2}{\left(\frac{1}{J_i}\hat{\sigma}_i^2 + \tilde{V}\right)^2} \left|\left(\bar{X}_i - \bar{X}\right)\left(Y_i - \bar{Y}\right)\right| > s, \left|\hat{V} - V\right| \leq \delta\right) \\
&\quad + \Pr\left(\left|\hat{V} - V\right| > \delta\right) \\
&\leq \Pr\left(\sqrt{n}\left|\hat{V} - V\right| \frac{1}{n}\sum_{i=1}^{n} \frac{\frac{1}{J_i}\hat{\sigma}_i^2}{\left(\frac{1}{J_i}\hat{\sigma}_i^2 + V - \delta\right)^2} \left|\left(\bar{X}_i - \bar{X}\right)\left(Y_i - \bar{Y}\right)\right| > s, \left|\hat{V} - V\right| \leq \delta\right) \\
&\quad + \Pr\left(\left|\hat{V} - V\right| > \delta\right) \\
&\leq \Pr\left(\sqrt{n}\left|\hat{V} - V\right| \frac{1}{n}\sum_{i=1}^{n} \frac{\frac{1}{J_i}\hat{\sigma}_i^2}{\left(\frac{1}{J_i}\hat{\sigma}_i^2 + V - \delta\right)^2} \left|\left(\bar{X}_i - \bar{X}\right)\left(Y_i - \bar{Y}\right)\right| > s\right) \\
&\quad + \Pr\left(\left|\hat{V} - V\right| > \delta\right)
\end{aligned}
$$

Then we only need to show

$$
\frac{1}{n}\sum_{i=1}^{n} \frac{\frac{1}{J_i}\hat{\sigma}_i^2}{\left(\frac{1}{J_i}\hat{\sigma}_i^2 + V - \delta\right)^2} \left|\left(\bar{X}_i - \bar{X}\right)\left(Y_i - \bar{Y}\right)\right| = o_p(1).
$$

This can be proved if for any integers $k_1, k_2 \in \{0, 1\}$,

$$
\frac{1}{n}\sum_{i=1}^{n} \frac{\frac{1}{J_i}\hat{\sigma}_i^2}{\left(\frac{1}{J_i}\hat{\sigma}_i^2 + V - \delta\right)^2} \left|\bar{X}_i\right|^{k_1} \left|Y_i\right|^{k_2} = o_p(1).
$$

For any $s > 0$, by Chebyshev's inequality,

$$
\begin{aligned}
&\Pr\left(\left|\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{1}{J_i}\hat{\sigma}_i^2}{\left(\frac{1}{J_i}\hat{\sigma}_i^2+V-\delta\right)^2}\left|\bar{X}_i\right|^{k_1}\left|Y_i\right|^{k_2}-\mathbb{E}\left[\frac{\frac{1}{J_i}\hat{\sigma}_i^2}{\left(\frac{1}{J_i}\hat{\sigma}_i^2+V-\delta\right)^2}\left|\bar{X}_i\right|^{k_1}\left|Y_i\right|^{k_2}\right]\right|>s\right)\\
&\leq\frac{\mathbb{E}\left[\left(\frac{\frac{1}{J_i}\hat{\sigma}_i^2}{\left(\frac{1}{J_i}\hat{\sigma}_i^2+V-\delta\right)^2}\right)^2\left|\bar{X}_i\right|^{2k_1}\left|Y_i\right|^{2k_2}\right]}{ns^2}\\
&\leq\frac{\frac{1}{(V-\delta)^2}\mathbb{E}\left[\left|\bar{X}_i\right|^{2k_1}\left(\alpha+\beta\theta_i+u_i\right)^{2k_2}\right]}{ns^2}\to 0.
\end{aligned}
$$

where the last limit follows from the independence and finite moments assumptions, as well as the fact from Lemma B.1 that $\mathbb{E}\left[\left|\bar{X}_i\right|^{2k_1}\right]\to\mathbb{E}\left[\left|\theta_i\right|^{2k_1}\right]$. Also, by the Cauchy-Schwarz inequality, from the property of $\left(\frac{1}{J_i}\hat{\sigma}_i^2\right)^2$ in Lemma B.1, we also have

$$
\begin{aligned}
\mathbb{E}\left[\frac{\frac{1}{J_i}\hat{\sigma}_i^2}{\left(\frac{1}{J_i}\hat{\sigma}_i^2+V-\delta\right)^2}\left|\bar{X}_i\right|^{k_1}\left|Y_i\right|^{k_2}\right]&=\mathbb{E}\left[\frac{\frac{1}{J_i}\hat{\sigma}_i^2}{\left(\frac{1}{J_i}\hat{\sigma}_i^2+V-\delta\right)^2}\right]\mathbb{E}\left[\left|\bar{X}_i\right|^{k_1}\left|Y_i\right|^{k_2}\right]\\
&\leq\frac{1}{(V-\delta)^2}\sqrt{\mathbb{E}\left[\left(\frac{1}{J_i}\hat{\sigma}_i^2\right)^2\right]\mathbb{E}\left[\left|\bar{X}_i\right|^{2k_1}\left|Y_i\right|^{2k_2}\right]}\to 0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{1}{J_i}\hat{\sigma}_i^2}{\left(\frac{1}{J_i}\hat{\sigma}_i^2+V-\delta\right)^2}\left|\left(\bar{X}_i-\bar{X}\right)\left(Y_i-\bar{Y}\right)\right|\\
&\leq\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{1}{J_i}\hat{\sigma}_i^2}{\left(\frac{1}{J_i}\hat{\sigma}_i^2+V-\delta\right)^2}\left|\bar{X}_iY_i\right|+\left|\bar{X}\right|\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{1}{J_i}\hat{\sigma}_i^2}{\left(\frac{1}{J_i}\hat{\sigma}_i^2+V-\delta\right)^2}\left|Y_i\right|\\
&\quad+\left|\bar{Y}\right|\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{1}{J_i}\hat{\sigma}_i^2}{\left(\frac{1}{J_i}\hat{\sigma}_i^2+V-\delta\right)^2}\left|\bar{X}_i\right|+\left|\bar{X}\right|\left|\bar{Y}\right|\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{1}{J_i}\hat{\sigma}_i^2}{\left(\frac{1}{J_i}\hat{\sigma}_i^2+V-\delta\right)^2}\\
&=o_p(1)+O_p(1)\cdot o_p(1)+O_p(1)\cdot o_p(1)+O_p(1)\cdot O_p(1)\cdot o_p(1)=o_p(1).
\end{aligned}
$$

Then the numerator part is shown.

Lastly, we show for the denominator part

$$\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_i-\bar{\hat{\theta}}\right)^2 \rightarrow_p \frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c}-\bar{\hat{\theta}}_c\right)^2.$$

Similarly, for simplicity of notations, we denote the whole denominator part as a function $g_n(t)$. Similarly, we have

$$g_n\left(\hat{V}\right)-g_n(V)=\left(\hat{V}-V\right)g_n'\left(\tilde{V}\right).$$

Here with a slight abuse of notation, $\tilde{V}$ satisfies $\left|\tilde{V}-V\right| \leq \left|\hat{V}-V\right|$. We finish the proof by showing

$$g_n'\left(\tilde{V}\right)=O_p(1).$$

Let

$$D_i := \frac{\frac{1}{J_i}\hat{\sigma}_i^2}{\left(\frac{1}{J_i}\hat{\sigma}_i^2+\tilde{V}\right)^2}.$$

$$G_i := \frac{\tilde{V}}{\frac{1}{J_i}\hat{\sigma}_i^2+\tilde{V}}.$$

$$\begin{aligned}
g_n'\left(\tilde{V}\right) &= \frac{2}{n}\sum_{i=1}^{n}\left[G_i\left(\bar{X}_i-\bar{X}\right)-\frac{1}{n}\sum_{k=1}^{n}G_k\left(\bar{X}_k-\bar{X}\right)\right]D_i\left(\bar{X}_i-\bar{X}\right)\\
&= \frac{2}{n}\sum_{i=1}^{n}G_iD_i\left(\bar{X}_i-\bar{X}\right)^2-2\left[\frac{1}{n}\sum_{i=1}^{n}G_i\left(\bar{X}_i-\bar{X}\right)\right]\left[\frac{1}{n}\sum_{i=1}^{n}D_i\left(\bar{X}_i-\bar{X}\right)\right].
\end{aligned}$$

Similarly, to show it's $O_p(1)$, it suffices to show for $k_1, k_2 \in \{0,1\}$, $k_3 \in \{0,1,2\}$,

$$\frac{1}{n}\sum_{i=1}^{n}G_i^{k_1}D_i^{k_2}\bar{X}_i^{k_3}=O_p(1).$$

For any $M > 0$, there exists $0 < \delta < V$, and

$$\begin{aligned}
&\Pr\left(\left|\frac{1}{n}\sum_{i=1}^{n} G_i^{k_1} D_i^{k_2} \bar{X}_i^{k_3}\right| > M\right) \\
\leq &\Pr\left(\frac{1}{n}\sum_{i=1}^{n} G_i^{k_1} D_i^{k_2} \left|\bar{X}_i\right|^{k_3} > M, \left|\tilde{V} - V\right| \leq \delta\right) + \Pr\left(\left|\tilde{V} - V\right| > \delta\right) \\
\leq &\Pr\left(\frac{1}{n}\sum_{i=1}^{n} \left(\frac{V-\delta}{\frac{1}{J_i}\hat{\sigma}_i^2 + V - \delta}\right)^{k_1} \left(\frac{\frac{1}{J_i}\hat{\sigma}_i^2}{\left(\frac{1}{J_i}\hat{\sigma}_i^2 + V - \delta\right)^2}\right)^{k_2} \left|\bar{X}_i\right|^{k_3} > M\right) + \Pr\left(\left|\tilde{V} - V\right| > \delta\right) \\
\leq &\Pr\left(\frac{1}{(V-\delta)^{k_2}} \frac{1}{n}\sum_{i=1}^{n} \left|\bar{X}_i\right|^{k_3} > M\right) + \Pr\left(\left|\tilde{V} - V\right| > \delta\right).
\end{aligned}$$

Because

$$\frac{1}{n}\sum_{i=1}^{n} \left|\bar{X}_i\right|^{k_3} = O_p(1),$$

therefore it is proved. □

Proof of Theorem 3.5. Firstly, for simplicity we abbreviate notations and denote $\hat{\beta}_{\mathrm{HE}}$ as $\hat{\beta}$, $\hat{\theta}_{i,\mathrm{HE}}$ as $\hat{\theta}_i$, and $\operatorname{Var}(\theta_i)$ as V. For the numerator, we have

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n} \left(\hat{\theta}_i - \bar{\hat{\theta}}\right)^2 \hat{u}_i^2 \\
= &\frac{1}{n}\sum_{i=1}^{n} \left(\hat{\theta}_i - \bar{\hat{\theta}}\right)^2 \left(Y_i - \bar{Y} - \hat{\beta}\left(\hat{\theta}_i - \bar{\hat{\theta}}\right)\right)^2 \\
= &\frac{1}{n}\sum_{i=1}^{n} \left(\hat{\theta}_i - \bar{\hat{\theta}}\right)^2 \left(Y_i - \bar{Y}\right)^2 - 2\hat{\beta}\frac{1}{n}\sum_{i=1}^{n} \left(\hat{\theta}_i - \bar{\hat{\theta}}\right)^3 \left(Y_i - \bar{Y}\right) \\
&+ \hat{\beta}^2 \frac{1}{n}\sum_{i=1}^{n} \left(\hat{\theta}_i - \bar{\hat{\theta}}\right)^4.
\end{aligned}$$

Then by Lemma C.1, in order to show

$$\frac{1}{n}\sum_{i=}^{n}\left(\hat{\theta}_i-\bar{\hat{\theta}}\right)^2\hat{u}_i^2\to\mathbb{E}\left[\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^2u_i^2\right],$$

it suffices to show for integers $2\leq k\leq 4$,

$$\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_i-\bar{\hat{\theta}}\right)^k\left(Y_i-\bar{Y}\right)^{4-k}\to_p\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c}-\bar{\hat{\theta}}_c\right)^k\left(Y_i-\bar{Y}\right)^{4-k}.$$

Because of the proof shown in Theorem 3.3, the above can be shown if we have $2\leq k\leq 4$,

$$\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_i-\bar{\hat{\theta}}\right)^kY_i^{4-k}\to_p\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c}-\bar{\hat{\theta}}_c\right)^kY_i^{4-k}$$

For simplicity of notations, we denote the left hand side as the function value of $f_n(t)$ at $t=\hat{V}$, and the right hand side as the function value at $t=V$. Then we have

$$f_n\left(\hat{V}\right)-f_n\left(V\right)=\left(\hat{V}-V\right)f_n'\left(\tilde{V}\right).$$

Here $\tilde{V}$ satisfies $\left|\tilde{V}-V\right|\leq\left|\hat{V}-V\right|$. Then we only need to show

$$f_n'\left(\tilde{V}\right)=O_p(1).$$

Let

$$D_i:=\frac{\frac{1}{J_i}\hat{\sigma}_i^2}{\left(\frac{1}{J_i}\hat{\sigma}_i^2+\tilde{V}\right)^2}.$$

$$G_i:=\frac{\tilde{V}}{\frac{1}{J_i}\hat{\sigma}_i^2+\tilde{V}}.$$

$$f_n'\left(\tilde{V}\right)$$
$$= \frac{k}{n}\sum_{i=1}^{n}\left[G_i\left(\bar{X}_i-\bar{X}\right)-\frac{1}{n}\sum_{k=1}^{n}G_k\left(\bar{X}_k-\bar{X}\right)\right]^{k-1}\left(D_i\left(\bar{X}_i-\bar{X}\right)-\frac{1}{n}\sum_{k=1}^{n}D_k\left(\bar{X}_k-\bar{X}\right)\right)Y_i^{4-k}$$

It suffices to show for $k_1 \in \{1,2,3\}$, $k_2 \in \{0,1\}$, $k_3 \in 1,2,3$,

$$\frac{1}{n}\sum_{i=1}^{n}G_i^{k_1}D_i^{k_2}\bar{X}_i^{k_3}Y_i^{4-k} = O_p(1).$$

For any $M > 0$, there exists $0 < \delta < V$, and

$$\Pr\left(\left|\frac{1}{n}\sum_{i=1}^{n}G_i^{k_1}D_i^{k_2}\bar{X}_i^{k_3}Y_i^{4-k}\right| > M\right)$$
$$\le \Pr\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{V-\delta}{\frac{1}{J_i}\hat{\sigma}_i^2+V-\delta}\right)^{k_1}\left(\frac{\frac{1}{J_i}\hat{\sigma}_i^2}{\left(\frac{1}{J_i}\hat{\sigma}_i^2+V-\delta\right)^2}\right)^{k_2}\left|\bar{X}_i\right|^{k_3}Y_i^{4-k} > M\right)+\Pr\left(\left|\tilde{V}-V\right|>\delta\right).$$
$$\le \Pr\left(\frac{1}{\left(V-\delta\right)^{k_2}}\frac{1}{n}\sum_{i=1}^{n}\left|\bar{X}_i\right|^{k_3}Y_i^{4-k} > M\right)+\Pr\left(\left|\tilde{V}-V\right|>\delta\right).$$

Because

$$\frac{1}{n}\sum_{i=1}^{n}\left|\bar{X}_i\right|^{k_3}Y_i^{4-k} = O_p(1),$$

therefore it is proved.

For the denominator, from the proof of Theorem 3.3 and Lemma 3.1 , we have

$$\left(\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_i-\bar{\hat{\theta}}\right)^2\right) \to_p V^2.$$

□

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Supplemental Appendix

B Lemmas and Proofs for Non-Normal Noise

Firstly, for simplicity we abbreviate notations and denote $\text{Var}(\theta_i)$ as V.

Remark B.1. Recall that the variance estimator is defined as

$$\hat{\sigma}_i^2 \coloneqq \frac{1}{J_i - 1} \sum_{j=1}^{J_i} \left(X_{i,j} - \bar{X}_i\right)^2 .$$

Then $\hat{\sigma}_i^2$ is unbiased for σ_i^2 and we have for its variance that

$$\begin{aligned} \text{Var}\left(\hat{\sigma}_i^2 \mid \sigma_i^2, J_i\right) &= \frac{1}{J_i}\mathbb{E}\left[\epsilon_{i,j}^4 \mid \sigma_i^2, J_i\right] - \frac{\sigma_i^2\left(J_i - 3\right)}{J_i\left(J_i - 1\right)} \\ &\le \frac{1}{J_i}K\sigma_i^4 - \frac{\sigma_i^2\left(J_i - 3\right)}{J_i\left(J_i - 1\right)} && \text{(Assumption 3.1.3)} \\ &\le \frac{1}{J_i}K\sigma_i^4, && \text{(Assumption 3.1.1)} \qquad (8) \end{aligned}$$

where the equality follows from e.g. O'Neill (2014) (Result 3, p. 284).

Lemma B.1. *Under Assumption 3.1, (3) and (4), we have:*

1. *Properties of $\bar{\epsilon}_i$, $\bar{X}_i$, $\hat{\sigma}_i^2$, c_i:*

   For any integer $k_1 \in \{1, 2\}$, $k_2 \ge 1$, we have

   (a) $\bar{\epsilon}_i = O_p(n^{-1/4})$, *and* $\mathbb{E}\left[|\bar{\epsilon}_i|^{k_1}\right] \to 0$.

   (b) $\bar{X}_i = \theta_i + O_p(n^{-1/4})$, *and* $\mathbb{E}\left[\left|\bar{X}_i\right|^{k_1}\right] \to \mathbb{E}\left[|\theta_i|^{k_1}\right]$.

   (c) $\hat{\sigma}_i^2 = \sigma_i^2 + O_p(n^{-1/4})$, *and* $\mathbb{E}\left[|\hat{\sigma}_i^2 - \sigma_i^2|^{k_1}\right] \to 0$.

   (d) $\mathbb{E}\left[\frac{1}{J_i}\hat{\sigma}_i^2\right] \to 0$, *and* $\mathbb{E}\left[\left(\frac{1}{J_i}\hat{\sigma}_i^2\right)^2\right] \to 0$.

   (e) $\mathbb{E}\left[\left(\frac{1}{J_i}\hat{\sigma}_i^2\right)^4\right] = o(n^{-1/2})$.

   (f) $c_i^{k_2} = 1 + O_p(n^{-1/2})$, *and* $\mathbb{E}\left[c_i^{k_2}\right] = 1 + O(n^{-1/2})$.

2. *Properties of sample moments of $\bar{\epsilon}_i$, c_i, θ_i:*

For any integer $k_1 \geq 0$[1], $0 \leq k_2 \leq 2$, *and* $k_3, k_4 \in \{0,1\}$, *we have*

$$\frac{1}{n}\sum_{i=1}^{n} c_i^{k_1}\left(\theta_i - k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2} = \mathbb{E}\left[\left(\theta_i - k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2}\right] + O_p(n^{-1/2}),$$
$$\frac{1}{n}\sum_{i=1}^{n} c_i^{k_1}\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)^{k_4}\bar{\epsilon}_i = o_p(n^{-1/2}),$$
$$\frac{1}{n}\sum_{i=1}^{n} c_i^{k_1}\left(\theta_i - k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2}\bar{\epsilon}_i^2 = O_p(n^{-1/2}).$$

3. *Properties of sample means of* θ_i, $\bar{X}_i$, $\hat{\sigma}_i^2$*:*

   (a) $\bar{\theta} = \mathbb{E}\left[\theta_i\right] + O_p(n^{-1/2})$.

   (b) $\bar{X} = \mathbb{E}\left[\theta_i\right] + O_p(n^{-1/2})$.

   (c) $\frac{1}{n}\sum_{i=1}^{n}\frac{1}{J_i}\hat{\sigma}_i^2 = O_p(n^{-1/2})$

4. *Properties of sample moments of* ϵ_i, c_i, u_i*:*

   For any integer $k \geq 0$, *we have*

   (a) $\frac{1}{n}\sum_{i=1}^{n} c_i^k u_i = O_p(n^{-1/2})$.

   (b) $\frac{1}{n}\sum_{i=1}^{n} c_i^k u_i \bar{\epsilon}_i = o_p(n^{-1/2})$.

Proof of Lemma B.1. Below we take any $s > 0$,

1. Properties of $\bar{\epsilon}_i$, $\bar{X}_i$, $\hat{\sigma}_i^2$, c_i:

   (a) By Markov's inequality,

$$\Pr\left(n^{1/4}\left|\bar{\epsilon}_i\right| > s\right) \leq \frac{\sqrt{n}\mathbb{E}\left[\bar{\epsilon}_i^2\right]}{s^2} = \frac{\sqrt{n}\mathbb{E}\left[\frac{1}{J_i}\right]\mathbb{E}\left[\sigma_i^2\right]}{s^2}.$$

   Since $\sqrt{n}\mathbb{E}\left[\frac{1}{J_i}\right] \to \kappa$, we have $\bar{\epsilon}_i = O_p(n^{-1/4})$.
   Since

$$\mathbb{E}\left[\bar{\epsilon}_i^2\right] = \mathbb{E}\left[\frac{1}{J_i}\right]\mathbb{E}\left[\sigma_i^2\right] \to 0,$$

[1] For statements like this, if $k = 0$ is on the exponent, it means that the term is 1.

combined with the Cauchy-Schwarz inequality, we have the rest of the results.

(b) This follows from 1a.

(c) By Markov's inequality and (8),

$$\Pr\left(n^{1/4}\left|\hat{\sigma}_i^2-\sigma_i^2\right|>s\right)\le\frac{\sqrt{n}\mathbb{E}\left[\left(\hat{\sigma}_i^2-\sigma_i^2\right)^2\right]}{s^2}$$
$$\le\frac{K\sqrt{n}\mathbb{E}\left[\frac{1}{J_i}\right]\mathbb{E}\left[\sigma_i^4\right]}{s^2}.$$

Due to (3), we have $\hat{\sigma}_i^2=\sigma_i^2+O_p(n^{-1/4})$.

Since by (8),

$$\mathbb{E}\left[\left(\hat{\sigma}_i^2-\sigma_i^2\right)^2\right]\le K\mathbb{E}\left[\frac{1}{J_i}\right]\mathbb{E}\left[\sigma_i^4\right]\to 0,$$

combined with the Cauchy-Schwarz inequality, we have the rest of the results.

(d) We have

$$\mathbb{E}\left[\frac{1}{J_i}\hat{\sigma}_i^2\right]=\mathbb{E}\left[\frac{1}{J_i}\right]\mathbb{E}\left[\sigma_i^2\right]\to 0,$$

$$\mathbb{E}\left[\left(\frac{1}{J_i}\hat{\sigma}_i^2\right)^2\right]\le\mathbb{E}\left[\frac{1}{J_i^2}\left(\frac{K\sigma_i^4}{J_i}+\sigma_i^4\right)\right]\to 0.\quad\text{(By (8))}$$

(e) From Theorem 2 of Angelova (2012), the fourth moment of $\hat{\sigma}_i^2$ is

$$\begin{aligned}\mathbb{E}\left[\left(\hat{\sigma}_i^2\right)^4 \mid J_i, \sigma_i^2\right] &= \mu_2^4 + \frac{6\mu_2^2(\mu_4 - \mu_2^2)}{J_i} + \frac{4\mu_6\mu_2 + 3\mu_4^2 - 18\mu_4\mu_2^2 - 24\mu_3^2\mu_2 + 23\mu_2^4}{J_i^2} \\ &+ \frac{\mu_8 - 4\mu_6\mu_2 - 24\mu_5\mu_3 - 3\mu_4^2 + 72\mu_4\mu_2^2 + 96\mu_3^2\mu_2 - 86\mu_2^4}{J_i^3} \\ &+ \frac{4(6\mu_6\mu_2 + 6\mu_4^2 - 39\mu_4\mu_2^2 - 40\mu_3^2\mu_2 + 45\mu_2^4)}{J_i^3(J_i - 1)} \\ &+ \frac{4(36\mu_4\mu_2^2 - 8\mu_5\mu_3 + 52\mu_3^2\mu_2 - 61\mu_2^4)}{J_i^3(J_i - 1)^2} \\ &+ \frac{8(\mu_4^2 - 6\mu_4\mu_2^2 - 12\mu_3^2\mu_2 + 15\mu_2^4)}{J_i^3(J_i - 1)^3},\end{aligned}$$

where $\mu_k, k = 1, ..., 8$ are the k-th raw moments of $\epsilon_{i,j} \mid \sigma_i^2$. By Assumption 3.1.3, we have $\mu_k \leq K\sigma_i^k$ for some constant K. Thus, by independence of Assumption 3.1.1 and (4), we have that

$$\sqrt{n}\mathbb{E}\left[\left(\frac{1}{J_i}\hat{\sigma}_i^2\right)^4\right] \to 0.$$

(f) By the mean value theorem,

$$\begin{aligned}c_i^{k_2} &= \left(\frac{V}{V + \frac{1}{J_i}\hat{\sigma}_i^2}\right)^{k_2} \\ &= 1 - k_2\frac{1}{VJ_i}\hat{\sigma}_i^2 + \frac{k_2\left(k_2+1\right)}{2}\frac{1}{\left(1+\omega_i\right)^{k_2+2}}\frac{1}{V^2J_i^2}\left(\hat{\sigma}_i^2\right)^2,\end{aligned}$$

where ω_i is between 0 and $\frac{1}{VJ_i}\hat{\sigma}_i^2$. Therefore,

$$\begin{aligned}\left|\sqrt{n}\mathbb{E}\left[1-c_i^{k_2}\right]\right| &\leq \frac{k_2}{V}\sqrt{n}\mathbb{E}\left[\frac{\sigma_i^2}{J_i}\right] \\ &\quad + \frac{k_2\left(k_2+1\right)}{2V^2}\sqrt{n}\mathbb{E}\left[\frac{1}{J_i^2}\left(\sigma_i^4+\frac{K}{J_i}\sigma_i^4\right)\right] && \text{(By (8))} \\ &\leq \frac{k_2}{V}\mathbb{E}\left[\sigma_i^2\right]\sqrt{n}\mathbb{E}\left[\frac{1}{J_i}\right] \\ &\quad + \frac{k_2\left(k_2+1\right)}{2V^2}\mathbb{E}\left[\sigma_i^4\right]\sqrt{n}\mathbb{E}\left[\frac{1}{J_i^2}\right] \\ &\quad + \frac{k_2\left(k_2+1\right)K}{2V^2}\mathbb{E}\left[\sigma_i^4\right]\sqrt{n}\mathbb{E}\left[\frac{1}{J_i^3}\right] && \text{(By Assumption 3.1.1)} \\ &\to \frac{k_2}{V}\mathbb{E}\left[\sigma_i^2\right]\kappa+0+0. && \text{(By (3) and (4))}\end{aligned}$$

Then we have the second result. By Markov's inequality, we have the first result.

2. By Chebyshev's inequality (k_2 can be replaced by k_4),

$$\begin{aligned}&\Pr\left(\sqrt{n}\left|\frac{1}{n}\sum_{i=1}^{n}c_i^{k_1}\left(\theta_i-k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2}-\mathbb{E}\left[c_i^{k_1}\left(\theta_i-k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2}\right]\right|>s\right) \\ &\leq \frac{\mathbb{E}\left[c_i^{2k_1}\left(\theta_i-k_3\mathbb{E}\left[\theta_i\right]\right)^{2k_2}\right]}{s^2} \leq \frac{\mathbb{E}\left[\left(\theta_i-k_3\mathbb{E}\left[\theta_i\right]\right)^{2k_2}\right]}{s^2}\end{aligned}$$

$$\begin{aligned}&\Pr\left(\sqrt{n}\left|\frac{1}{n}\sum_{i=1}^{n}c_i^{k_1}\left(\theta_i-k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2}\bar{\epsilon}_i-\mathbb{E}\left[c_i^{k_1}\left(\theta_i-k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2}\bar{\epsilon}_i\right]\right|>s\right) \\ &\leq \frac{\mathbb{E}\left[c_i^{2k_1}\left(\theta_i-k_3\mathbb{E}\left[\theta_i\right]\right)^{2k_2}\bar{\epsilon}_i^2\right]}{s^2} \leq \frac{\mathbb{E}\left[\left(\theta_i-k_3\mathbb{E}\left[\theta_i\right]\right)^{2k_2}\right]\mathbb{E}\left[\frac{1}{J_i}\right]\mathbb{E}\left[\sigma_i^2\right]}{s^2} \to 0,\end{aligned}$$

$$
\begin{aligned}
&\Pr\left(\sqrt{n}\left|\frac{1}{n}\sum_{i=1}^{n} c_i^{k_1}\left(\theta_i - k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2}\bar{\epsilon}_i^2 - \mathbb{E}\left[c_i^{k_1}\left(\theta_i - k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2}\bar{\epsilon}_i^2\right]\right| > s\right)\\
\leq{}& \frac{\mathbb{E}\left[c_i^{2k_1}\left(\theta_i - k_3\mathbb{E}\left[\theta_i\right]\right)^{2k_2}\bar{\epsilon}_i^4\right]}{s^2}\\
\leq{}& \frac{\mathbb{E}\left[\left(\theta_i - k_3\mathbb{E}\left[\theta_i\right]\right)^{2k_2}\right]\left(\mathbb{E}\left[\frac{1}{J_i^3}\right]\mathbb{E}\left[K\sigma_i^4\right] + 3\mathbb{E}\left[\frac{J_i-1}{J_i^3}\right]\mathbb{E}\left[\sigma_i^4\right]\right)}{s^2} \to 0.
\end{aligned}
$$

Meanwhile, by independence from Assumption 3.1.3,

$$
\begin{aligned}
&\mathbb{E}\left[c_i^{k_1}\left(\theta_i - k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2}\right] = \mathbb{E}\left[c_i^{k_1}\right]\mathbb{E}\left[\left(\theta_i - k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2}\right]\\
={}& \left(1 + O(n^{-1/2})\right)\mathbb{E}\left[\left(\theta_i - k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2}\right] && \text{(By 1f)}\\
={}& \mathbb{E}\left[\left(\theta_i - k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2}\right] + O(n^{-1/2}),
\end{aligned}
$$

by the mean value theorem ($k_4 = 0$) and (8),

$$
\begin{aligned}
\left|\sqrt{n}\mathbb{E}\left[c_i^{k_1}\bar{\epsilon}_i\right]\right| &\leq 0 + \frac{k_1}{V}\sqrt{n}\mathbb{E}\left[\frac{1}{J_i}\hat{\sigma}_i^2\bar{\epsilon}_i\right]\\
&\quad + \frac{k_1\left(k_1+1\right)}{2V^2}\sqrt{n}\mathbb{E}\left[\frac{1}{J_i^2}\left(\hat{\sigma}_i^2\right)^2\bar{\epsilon}_i\right]\\
&\leq \frac{k_1}{V}\sqrt{\sqrt{n}\mathbb{E}\left[\frac{1}{J_i^2}\left(\sigma_i^4 + \frac{K}{J_i}\sigma_i^4\right)\right]\sqrt{n}\mathbb{E}\left[\frac{1}{J_i}\sigma_i^2\right]}\\
&\quad + \frac{k_1\left(k_1+1\right)}{2V^2}\sqrt{\sqrt{n}\mathbb{E}\left[\frac{1}{J_i^4}\left(\hat{\sigma}_i^2\right)^4\right]\sqrt{n}\mathbb{E}\left[\frac{1}{J_i}\sigma_i^2\right]} && \text{(Cauchy-Schwarz)}\\
&\to 0,
\end{aligned}
$$

where the last step follows from independence of Assumption 3.1.1, (3), (4) and 1e,

by the independence in Assumption 3.1.3 ($k_4 = 1$),

$$
\mathbb{E}\left[c_i^{k_1}\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)\bar{\epsilon}_i\right] = \mathbb{E}\left[\mathbb{E}\left(c_i^{k_1}\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)\bar{\epsilon}_i\middle|\epsilon\right)\right] = 0,
$$

and also by Assumption 3.1.3,

$$\left|\sqrt{n}\mathbb{E}\left[c_i^{k_1}\left(\theta_i - k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2}\bar{\epsilon}_i^2\right]\right| \leq \mathbb{E}\left[\left|\left(\theta_i - k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2}\right|\right]\sqrt{n}\mathbb{E}\left[\frac{1}{J_i}\right]\mathbb{E}\left[\sigma_i^2\right],$$

where $\sqrt{n}\mathbb{E}\left[\frac{1}{J_i}\right] \to \kappa$. Therefore we have the results.

3. (a) Follows from the finite fourth moment and the inequality below for any $s > 0$,

$$\Pr\left(\left|\sqrt{n}\left(\bar{\theta} - \mathbb{E}\left[\theta_i\right]\right)\right| > s\right) \leq \frac{\mathbb{E}\left[\theta_i^2\right]}{s^2}. \qquad \text{(Chebyshev's inequality)}$$

   (b) Follows from the property of $\bar{\theta}$ and the second property of 2:

$$\bar{X} = \bar{\theta} + \bar{\epsilon} = \mathbb{E}\left[\theta_i\right] + O_p(n^{-1/2}) + o_p(n^{-1/2}) = \mathbb{E}\left[\theta_i\right] + O_p(n^{-1/2}).$$

   (c) By Chebyshev's inequality and (8), for any $s > 0$,

$$\Pr\left(\sqrt{n}\left|\frac{1}{n}\sum_{i=1}^{n}\frac{1}{J_i}\hat{\sigma}_i^2 - \mathbb{E}\left[\frac{1}{J_i}\hat{\sigma}_i^2\right]\right| > s\right) \leq \frac{\mathbb{E}\left[\frac{1}{J_i^2}\left(\frac{K\sigma_i^4}{J_i} + \sigma_i^4\right)\right]}{s^2} \to 0,$$

   and also

$$\sqrt{n}\mathbb{E}\left[\frac{1}{J_i}\hat{\sigma}_i^2\right] = \sqrt{n}\mathbb{E}\left[\frac{1}{J_i}\right]\mathbb{E}\left[\sigma_i^2\right] \to \kappa\mathbb{E}\left[\sigma_i^2\right].$$

   Thus

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{J_i}\hat{\sigma}_i^2 = O_p(n^{-1/2}).$$

4. (a) Since $u_i \perp\!\!\!\perp \epsilon_{i,j} \mid \theta_i$, by Chebyshev's inequality,

$$\Pr\left(\sqrt{n}\left|\frac{1}{n}\sum_{i=1}^{n}c_i^k u_i\right| > s\right) \leq \frac{\mathbb{E}\left[c_i^{2k}u_i^2\right]}{s^2} \leq \frac{\sigma_u^2}{s^2}$$

Thus

$$\frac{1}{n}\sum_{i=1}^{n} c_i^k u_i = O_p(n^{-1/2}).$$

(b) By Chebyshev's inequality,

$$\Pr\left(\sqrt{n}\left|\frac{1}{n}\sum_{i=1}^{n} c_i^k u_i \bar{\epsilon}_i\right| > s\right) \le \frac{\mathbb{E}\left[c_i^{2k} u_i^2 \bar{\epsilon}_i^2\right]}{s^2} \le \frac{\sigma_u^2}{s^2}\mathbb{E}\left[\frac{1}{J_i}\right]\mathbb{E}\left[\sigma_i^2\right] \to 0.$$

Thus

$$\frac{1}{n}\sum_{i=1}^{n} c_i^k u_i \bar{\epsilon}_i = o_p(n^{-1/2}).$$

□

Lemma B.2. *Under Assumption 3.1, (3) and (4), we have*

$$\frac{1}{n}\sum_{i=1}^{n} c_i^2 \bar{\epsilon}_i^2 = \frac{1}{n}\sum_{i=1}^{n} c_i^2 \frac{1}{J_i}\hat{\sigma}_i^2 + o_p(n^{-1/2}).$$

Proof of Lemma B.2. First, notice that the difference

$$\begin{aligned}
\bar{\epsilon}_i^2 - \frac{1}{J_i}\hat{\sigma}_i^2 &= \bar{\epsilon}_i^2 - \frac{1}{J_i\left(J_i-1\right)}\sum_{j=1}^{J_i}\epsilon_{i,j}^2 + \frac{1}{J_i-1}\bar{\epsilon}_i^2 \\
&= \frac{1}{J_i\left(J_i-1\right)}\left(\sum_{j=1}^{J_i}\epsilon_{i,j}\right)^2 - \frac{1}{J_i\left(J_i-1\right)}\sum_{j=1}^{J_i}\epsilon_{i,j}^2 \\
&= \frac{2}{J_i\left(J_i-1\right)}\sum_{k\le j}\epsilon_{i,k}\epsilon_{i,j}.
\end{aligned}$$

Then the properties of the difference are

$$\begin{aligned}
&\mathbb{E}\left[\bar{\epsilon}_i^2 - \frac{1}{J_i}\hat{\sigma}_i^2\right] = 0, \\
&\mathbb{E}\left[\left(\bar{\epsilon}_i^2 - \frac{1}{J_i}\hat{\sigma}_i^2\right)^2 \mid J_i, \sigma_i^2\right] = \frac{2\sigma_i^4}{J_i\left(J_i-1\right)}.
\end{aligned}$$

For c_i^2, by the mean value theorem,

$$\begin{aligned} c_i^2 &= \left(\frac{V}{V+\frac{1}{J_i}\hat{\sigma}_i^2}\right)^2 \\ &= 1 - 2\frac{1}{VJ_i}\hat{\sigma}_i^2 + 3\frac{1}{(1+\omega_i)^4}\frac{1}{V^2J_i^2}\left(\hat{\sigma}_i^2\right)^2, \end{aligned}$$

where ω_i is between 0 and $\frac{1}{VJ_i}\hat{\sigma}_i^2$. Therefore,

$$\begin{aligned} &\left|\sqrt{n}\mathbb{E}\left[c_i^2\left(\bar{\epsilon}_i^2 - \frac{1}{J_i}\hat{\sigma}_i^2\right)\right]\right| \\ &\leq \left|\sqrt{n}\mathbb{E}\left[\left(\bar{\epsilon}_i^2 - \frac{1}{J_i}\hat{\sigma}_i^2\right)\right]\right| + \frac{2}{V}\left|\sqrt{n}\mathbb{E}\left[\frac{1}{J_i}\hat{\sigma}_i^2\left(\bar{\epsilon}_i^2 - \frac{1}{J_i}\hat{\sigma}_i^2\right)\right]\right| \\ &\quad + \frac{3}{V^2}\left|\sqrt{n}\mathbb{E}\left[\frac{1}{(1+\omega_i)^4}\frac{1}{J_i^2}\left(\hat{\sigma}_i^2\right)^2\left(\bar{\epsilon}_i^2 - \frac{1}{J_i}\hat{\sigma}_i^2\right)\right]\right| \\ &\leq 0 + \frac{2}{V}\sqrt{n\mathbb{E}\left[\frac{1}{J_i^2}\left(\hat{\sigma}_i^2\right)^2\right]\mathbb{E}\left[\left(\bar{\epsilon}_i^2 - \frac{1}{J_i}\hat{\sigma}_i^2\right)^2\right]} \\ &\quad + \frac{3}{V^2}\sqrt{n\mathbb{E}\left[\frac{1}{J_i^4}\left(\hat{\sigma}_i^2\right)^4\right]\mathbb{E}\left[\left(\bar{\epsilon}_i^2 - \frac{1}{J_i}\hat{\sigma}_i^2\right)^2\right]} && \text{(Cauchy-Schwarz)} \\ &\leq \frac{2}{V}\sqrt{\sqrt{n}\mathbb{E}\left[\frac{1}{J_i^2}\left(\frac{K}{J_i}\sigma_i^4 + \sigma_i^4\right)\right]\sqrt{n}\mathbb{E}\left[\frac{2\sigma_i^4}{J_i\left(J_i-1\right)}\right]} \\ &\quad + \frac{3}{V^2}\sqrt{\sqrt{n}\mathbb{E}\left[\frac{1}{J_i^4}\left(\hat{\sigma}_i^2\right)^4\right]\sqrt{n}\mathbb{E}\left[\frac{2\sigma_i^4}{J_i\left(J_i-1\right)}\right]}. && \text{(By properties of the difference)} \end{aligned}$$

Since we have independence of $\sigma_i^2 \perp\!\!\!\perp J_i$ from Assumption 3.1.1, (3) (4), and that

$$\sqrt{n}\mathbb{E}\left[\frac{1}{J_i^4}\left(\hat{\sigma}_i^2\right)^4\right] \to 0,$$

from Lemma B.1.1e, the above converges to 0. Thus, the expectation

$$\sqrt{n}\mathbb{E}\left[c_i^2\left(\bar{\epsilon}_i^2 - \frac{1}{J_i}\hat{\sigma}_i^2\right)\right] \to 0.$$

Next, we show the convergence to expectations by Chebyshev's inequality:

$$\Pr\left(\sqrt{n}\left|\frac{1}{n}\sum_{i=1}^{n} c_i^2\left(\bar{\epsilon}_i^2 - \frac{1}{J_i}\hat{\sigma}_i^2\right) - \mathbb{E}\left[c_i^2\left(\bar{\epsilon}_i^2 - \frac{1}{J_i}\hat{\sigma}_i^2\right)\right]\right| > s\right)$$
$$\leq \frac{\mathbb{E}\left[c_i^4\left(\bar{\epsilon}_i^2 - \frac{1}{J_i}\hat{\sigma}_i^2\right)^2\right]}{s^2} \leq \frac{\mathbb{E}\left[\left(\bar{\epsilon}_i^2 - \frac{1}{J_i}\hat{\sigma}_i^2\right)^2\right]}{s^2} = \frac{\mathbb{E}\left[\frac{2\sigma_i^4}{J_i(J_i-1)}\right]}{s^2} \to 0.$$

And the proof is complete. □

Recall that we define in the beginning of this section $V \coloneqq \mathrm{Var}\left(\theta_i\right)$ and in Section 3.3

$$\hat{V} \coloneqq \frac{1}{n}\sum_{i=1}^{n}\left(\bar{X}_i - \bar{X}\right)^2 - \frac{n-1}{n^2}\sum_{i=1}^{n}\frac{1}{J_i}\hat{\sigma}_i^2.$$

The following lemma shows that $\hat{V}$ converges to V at $\sqrt{n}$-rate.

Lemma B.3. *Under Assumption 3.1, (3) and (4), we have*

$$\hat{V} = V + O_p(n^{-1/2}).$$

Proof of Lemma B.3. By Lemma B.1,

$$\begin{aligned}
\hat{V} &= \frac{1}{n}\sum_{i=1}^{n}\left(\bar{X}_i - \bar{X}\right)^2 - \frac{n-1}{n^2}\sum_{i=1}^{n}\frac{1}{J_i}\hat{\sigma}_i^2 \\
&= \frac{1}{n}\sum_{i=1}^{n}\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)^2 + \frac{1}{n}\sum_{i=1}^{n}\bar{\epsilon}_i^2 - \left(\bar{\theta} - \mathbb{E}\left[\theta_i\right]\right)^2 - \bar{\epsilon}^2 + \frac{2}{n}\sum_{i=1}^{n}\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)\bar{\epsilon}_i \\
&\quad - 2\left(\bar{\theta} - \mathbb{E}\left[\theta_i\right]\right)\bar{\epsilon} - \frac{n-1}{n^2}\sum_{i=1}^{n}\frac{1}{J_i}\hat{\sigma}_i^2 \\
&= V + O_p(n^{-1/2}) + O_p(n^{-1/2}) - O_p(n^{-1}) - o_p(n^{-1}) + o_p(n^{-1/2}) \\
&\quad - o_p(n^{-1}) - O_p(n^{-1/2}) \\
&= V + O_p(n^{-1/2}).
\end{aligned}$$

□

We prove the common-weight result here.

Proof of Proposition 3.6. Firstly, for simplicity we abbreviate notations and denote $\hat{\beta}_{\text{CW}}$ as $\hat{\beta}$, and $\text{Var}(\theta_i)$ as V.

$$\sqrt{n}\left(\hat{\beta}-\beta\right) = \hat{V}^{-1}\left(\beta\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\bar{X}_i-\bar{X}\right)\left(\theta_i-\bar{\theta}\right)-\beta\sqrt{n}\hat{V}+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\bar{X}_i-\bar{X}\right)(u_i-\bar{u})\right)$$
$$= \frac{\beta\sqrt{n}T_{1,n}-\beta\sqrt{n}T_{2.n}+\sqrt{n}T_{3,n}}{T_{2,n}}.$$

Firstly, from Lemma B.1 and from the proof of Lemma B.2, we have for the denominator,

$$T_{2,n} = \frac{1}{n}\sum_{i=1}^{n}\left(\bar{X}_i-\bar{X}\right)^2-\frac{1}{n}\sum_{i=1}^{n}\frac{1}{J_i}\hat{\sigma}_i^2+o_p(n^{-1/2})$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left(\bar{X}_i-\bar{X}\right)^2-\frac{1}{n}\sum_{i=1}^{n}\bar{\epsilon}_i^2+o_p(n^{-1/2}).$$

From Lemma B.3,

$$T_{2,n} = V+O_p(n^{-1/2}).$$

For the numerator terms, by properties from Lemma B.1,

$$\sqrt{n}T_{1,n} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^2+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\bar{\epsilon}_i\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)$$
$$-\sqrt{n}\left(\bar{\theta}-\mathbb{E}\left[\theta_i\right]\right)^2-\sqrt{n}\left(\bar{\theta}-\mathbb{E}\left[\theta_i\right]\right)\bar{\epsilon}$$
$$= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^2+o_p(1),$$

$$\sqrt{n}T_{3,n} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)u_i+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\bar{\epsilon}_i u_i$$
$$-\sqrt{n}\left(\bar{\theta}-\mathbb{E}\left[\theta_i\right]\right)\bar{u}-\sqrt{n}\bar{\epsilon}\bar{u}$$
$$= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)u_i+o_p(1).$$

Combined with the proof of Lemma B.3, we have

$$\begin{aligned}
\sqrt{n}T_{2,n} &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^2+\bar{\epsilon}_i^2\right]-\sqrt{n}\left(\bar{\theta}-\mathbb{E}\left[\theta_i\right]\right)^2-\sqrt{n}\bar{\epsilon}^2+\frac{2}{\sqrt{n}}\sum_{i=1}^{n}\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)\bar{\epsilon}_i \\
&\quad -2\sqrt{n}\left(\bar{\theta}-\mathbb{E}\left[\theta_i\right]\right)\bar{\epsilon}-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\bar{\epsilon}_i^2 \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^2+o_p(1).
\end{aligned}$$

Therefore, the numerator is

$$\begin{aligned}
&\beta\sqrt{n}T_{1,n}-\beta\sqrt{n}T_{2,n}+\sqrt{n}T_{3,n} \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)u_i+o_p(1).
\end{aligned}$$

Then applying the central limit theorem combined with the denominator, by Slutsky's theorem,

$$\sqrt{n}\left(\hat{\beta}-\beta\right)=\frac{\beta\sqrt{n}T_{1,n}-\beta\sqrt{n}T_{2,n}+\sqrt{n}T_{3,n}}{T_{2,n}}\to_d N(0,\frac{\mathbb{E}\left[u_i^2\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^2\right]}{V^2}).$$

□

C Lemmas and Proofs for Inference

Firstly, for simplicity we abbreviate notations and denote $\hat{\beta}_{c,\mathrm{HE}}$ as $\hat{\beta}_c$, $\hat{\beta}_{\mathrm{HE}}$ as $\hat{\beta}$, $\hat{\theta}_{i,c,\mathrm{HE}}$ as $\hat{\theta}_{i,c}$, $\hat{\theta}_{i,\mathrm{HE}}$ as $\hat{\theta}_i$ and $\mathrm{Var}\left(\theta_i\right)$ as V.

Lemma C.1. *Under Assumption 3.2,*

$$\frac{1}{n}\sum_{i-1}^{n}\left(\hat{\theta}_{i,c}-\bar{\hat{\theta}}_c\right)^2\hat{u}_{i,c}^2\to_p\mathbb{E}\left[u_i^2\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^2\right].$$

Proof of Lemma C.1. Combining Lemma C.2, C.3, C.4, C.5, we have

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c}-\bar{\hat{\theta}}_c\right)^2\hat{u}_{i,c}^2 \\
=&\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c}-\bar{\hat{\theta}}_c\right)^2\left(Y_i-\bar{Y}-\hat{\beta}_c\left(\hat{\theta}_{i,c}-\bar{\hat{\theta}}_c\right)\right)^2 \\
=&\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c}-\bar{\hat{\theta}}_c\right)^2\left(Y_i-\bar{Y}\right)^2-2\hat{\beta}_c\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c}-\bar{\hat{\theta}}_c\right)^3\left(Y_i-\bar{Y}\right) \\
&+\hat{\beta}_c^2\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c}-\bar{\hat{\theta}}_c\right)^4. \\
=&\beta^2\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c}-\bar{\hat{\theta}}_c\right)^2\left(\theta_i-\bar{\theta}\right)^2-2\beta\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c}-\bar{\hat{\theta}}_c\right)^2\left(\theta_i-\bar{\theta}\right)\left(u_i-\bar{u}\right) \\
&+\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c}-\bar{\hat{\theta}}_c\right)^2\left(u_i-\bar{u}\right)^2 \\
&-2\hat{\beta}_c\beta\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c}-\bar{\hat{\theta}}_c\right)^3\left(\theta_i-\bar{\theta}\right)+2\hat{\beta}_c\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c}-\bar{\hat{\theta}}_c\right)^3\left(u_i-\bar{u}\right) \\
&+\hat{\beta}_c^2\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c}-\bar{\hat{\theta}}_c\right)^4 \\
=&\beta^2\mathbb{E}\left[\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^4\right]-2\beta\mathbb{E}\left[\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^3u_i\right]+\mathbb{E}\left[\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^2u_i^2\right] \\
&-2\beta^2\mathbb{E}\left[\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^4\right]+2\beta\mathbb{E}\left[\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^3u_i\right]+\beta^2\mathbb{E}\left[\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^4\right]+o_p(1) \\
=&\mathbb{E}\left[u_i^2\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^2\right]+o_p(1).
\end{aligned}$$

□

Lemma C.2. *Under Assumption 3.2,*

$$\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c}-\bar{\hat{\theta}}_c\right)^4\to_p\mathbb{E}\left[\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^4\right].$$

Proof of Lemma C.2. For any $s>0$ and integers $k_1\geq 0$, $0\leq k_2,k_3\leq 4$,

$$\Pr\left(\left|\frac{1}{n}\sum_{i=1}^{n}c_i^{k_1}\bar{X}_i^{k_2}-\mathbb{E}\left[c_i^{k_1}\bar{X}_i^{k_2}\right]\right|>s\right)\leq\frac{\mathbb{E}\left[c_i^{2k_1}\bar{X}_i^{2k_2}\right]}{ns^2}\leq\frac{\mathbb{E}\left[\bar{X}_i^{2k_2}\right]}{ns^2}\to 0.$$

$$\mathbb{E}\left[c_i^{k_1}\theta_i^{k_3}\right] = \mathbb{E}\left[c_i^{k_1}\right]\mathbb{E}\left[\theta_i^{k_3}\right] \to \mathbb{E}\left[\theta_i^{k_3}\right]. \qquad \text{(By Lemma B.1 1f)}$$

For $k_4 = 4$, by independence and (8),

$$\begin{aligned}\left|\mathbb{E}\left[c_i^{k_1}\theta_i^{k_3}\bar{\epsilon}_i^{k_4}\right]\right| \leq \mathbb{E}\left[\left|\theta_i^{k_3}\right|\right]\mathbb{E}\left[\bar{\epsilon}_i^{k_4}\right] &= \mathbb{E}\left[\left|\theta_i^{k_3}\right|\right]\mathbb{E}\left[\frac{1}{J_i^3}\mathbb{E}\left[\epsilon_{i,j}^4 \mid J_i, \sigma_i^2\right] + \frac{3\left(J_i-1\right)}{J_i^3}\sigma_i^2\right] \\ &\leq \mathbb{E}\left[\left|\theta_i^{k_3}\right|\right]\mathbb{E}\left[\frac{1}{J_i^3}K\sigma_i^4 + \frac{3\left(J_i-1\right)}{J_i^3}\sigma_i^2\right] \\ &= \mathbb{E}\left[\left|\theta_i^{k_3}\right|\right]\left(\mathbb{E}\left[\frac{1}{J_i^3}\right]K\mathbb{E}\left[\sigma_i^4\right] + \mathbb{E}\left[\frac{3\left(J_i-1\right)}{J_i^3}\right]\mathbb{E}\left[\sigma_i^2\right]\right) \to 0.\end{aligned}$$

Then by Jensen's inequality, for $1 \leq k_4 \leq 4$,

$$\left|\mathbb{E}\left[c_i^{k_1}\theta_i^{k_3}\bar{\epsilon}_i^{k_4}\right]\right| \to 0.$$

Therefore for $0 \leq k_2 \leq 4$,

$$\begin{aligned}&\mathbb{E}\left[c_i^{k_1}\bar{X}_i^{k_2}\right] \\ &= \sum_{k_4=0}^{k_2}\binom{k_2}{k_4}\mathbb{E}\left[c_i^{k_1}\theta_i^{k_2-k_4}\bar{\epsilon}_i^{k_4}\right] \\ &\to \mathbb{E}\left[\theta_i^{k_2}\right].\end{aligned}$$

So for $0 \leq k_2 \leq 4$,

$$\frac{1}{n}\sum_{i=1}^{n} c_i^{k_1}\bar{X}_i^{k_2} \to_p \mathbb{E}\left[\theta_i^{k_2}\right].$$

Therefore for $1 \leq k \leq 4$, $k_2 \geq 0$,

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\left(c_i\bar{X}_i - \frac{1}{n}\sum_{i=1}^{n}c_i\bar{X}_i\right)^k \\
&= \sum_{k_1=0}^{k}\binom{k}{k_1}\left(-\frac{1}{n}\sum_{i=1}^{n}c_i\bar{X}_i\right)^{k-k_1}\frac{1}{n}\sum_{i=1}^{n}c_i^{k_1}\bar{X}_i^{k_1} \\
&= \sum_{k_1=0}^{k}\binom{k}{k_1}\left(-\mathbb{E}\left[\theta_i\right]\right)^{k-k_1}\mathbb{E}\left[\theta_i^{k_1}\right] + o_p(1) \\
&= \mathbb{E}\left[\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)^k\right] + o_p(1).
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\left(c_i\bar{X}_i - \frac{1}{n}\sum_{i=1}^{n}c_i\bar{X}_i\right)^k c_i^{k_2} \\
&= \sum_{k_1=0}^{k}\binom{k}{k_1}\left(-\frac{1}{n}\sum_{i=1}^{n}c_i\bar{X}_i\right)^{k-k_1}\frac{1}{n}\sum_{i=1}^{n}c_i^{k_1+k_2}\bar{X}_i^{k_1} \\
&= \mathbb{E}\left[\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)^k\right] + o_p(1).
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\left(c_i\bar{X}_i - \frac{1}{n}\sum_{i=1}^{n}c_i\bar{X}_i\right)^k \left(c_i - \bar{c}\right)^{k_2} \\
&= \sum_{k_1=0}^{k}\binom{k}{k_1}\left(-\frac{1}{n}\sum_{i=1}^{n}c_i\bar{X}_i\right)^{k-k_1}\frac{1}{n}\sum_{i=1}^{n}c_i^{k_1+k_2}\bar{X}_i^{k_1}\left(c_i - \bar{c}\right)^{k_2} \\
&= \sum_{k_1=0}^{k}\binom{k}{k_1}\left(-\frac{1}{n}\sum_{i=1}^{n}c_i\bar{X}_i\right)^{k-k_1}\sum_{t=0}^{k_2}\binom{k_2}{t}\left(-\bar{c}\right)^{k_2-t}\frac{1}{n}\sum_{i=1}^{n}c_i^{k_1+k_2+t}\bar{X}_i^{k_1} \\
&= \sum_{k_1=0}^{k}\binom{k}{k_1}\left(-\mathbb{E}\left[\theta_i\right]\right)^{k-k_1}\sum_{t=0}^{k_2}\binom{k_2}{t}\left(-1\right)^{k_2-t}\mathbb{E}\left[\theta_i^{k_1}\right] + o_p(1) \\
&= o_p(1).
\end{aligned}$$

The above also applies to $k = 0$.

Therefore,

$$
\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c}-\bar{\hat{\theta}}_{c}\right)^{4}\\
&=\frac{1}{n}\sum_{i=1}^{n}\left(c_i\left(\bar{X}_i-\bar{X}\right)-\frac{1}{n}\sum_{i=1}^{n}\left(c_i\left(\bar{X}_i-\bar{X}\right)\right)\right)^{4}\\
&=\frac{1}{n}\sum_{i=1}^{n}\left(c_i\bar{X}_i-\frac{1}{n}\sum_{i=1}^{n}c_i\bar{X}_i-\bar{X}\left(c_i-\bar{c}\right)\right)^{4}\\
&=\frac{1}{n}\sum_{i=1}^{n}\left(c_i\bar{X}_i-\frac{1}{n}\sum_{i=1}^{n}c_i\bar{X}_i\right)^{4}-4\bar{X}\frac{1}{n}\sum_{i=1}^{n}\left(c_i\bar{X}_i-\frac{1}{n}\sum_{i=1}^{n}c_i\bar{X}_i\right)^{3}\left(c_i-\bar{c}\right)\\
&\quad+6\bar{X}^2\frac{1}{n}\sum_{i=1}^{n}\left(c_i\bar{X}_i-\frac{1}{n}\sum_{i=1}^{n}c_i\bar{X}_i\right)^{2}\left(c_i-\bar{c}\right)^2-4\bar{X}^3\frac{1}{n}\sum_{i=1}^{n}\left(c_i\bar{X}_i-\frac{1}{n}\sum_{i=1}^{n}c_i\bar{X}_i\right)\left(c_i-\bar{c}\right)^3\\
&\quad+\bar{X}^4\frac{1}{n}\sum_{i=1}^{n}\left(c_i-\bar{c}\right)^4.\\
&=\mathbb{E}\left[\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^4\right]+o_p(1).
\end{aligned}
$$

□

Lemma C.3. *Under Assumption 3.2,*

$$\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c}-\bar{\hat{\theta}}_{c}\right)^{k}\left(\theta_i-\bar{\theta}\right)^{4-k}\to_p\mathbb{E}\left[\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^4\right].$$

Proof of Lemma C.3. For any $s>0$ and integers $k_1\geq 0$, $0\leq k_2\leq 4$, $k_2+k_3\leq 4$, by Chebyshev's inequality,

$$\Pr\left(\left|\frac{1}{n}\sum_{i=1}^{n}c_i^{k_1}\bar{X}_i^{k_2}\theta_i^{k_3}-\mathbb{E}\left[c_i^{k_1}\bar{X}_i^{k_2}\theta_i^{k_3}\right]\right|>s\right)\leq\frac{\mathbb{E}\left[c_i^{2k_1}\bar{X}_i^{2k_2}\theta_i^{2k_3}\right]}{ns^2}\leq\frac{\mathbb{E}\left[\bar{X}_i^{2k_2}\theta_i^{2k_3}\right]}{ns^2}\to 0.$$

For $0 \le k_2 \le 4$, $k_2 + k_3 \le 4$,

$$\begin{aligned}
&\mathbb{E}\left[c_i^{k_1}\bar{X}_i^{k_2}\theta_i^{k_3}\right] \\
&= \sum_{k_4=0}^{k_2}\binom{k_2}{k_4}\mathbb{E}\left[c_i^{k_1}\theta_i^{k_2-k_4+k_3}\bar{\epsilon}_i^{k_4}\right] \\
&\to \mathbb{E}\left[\theta_i^{k_2+k_3}\right]. && \text{(By Lemma B.1 1f)}
\end{aligned}$$

Therefore, similar to the proof of Lemma C.2,

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c}-\bar{\hat{\theta}}_c\right)^k\left(\theta_i-\bar{\theta}\right)^{4-k} \\
&= \sum_{k_1=0}^{4-k}\binom{4-k}{k_1}\bar{\theta}^{4-k-k_1}\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c}-\bar{\hat{\theta}}_c\right)^k\theta_i^{k_1} \\
&= \sum_{k_1=0}^{4-k}\binom{4-k}{k_1}\left(\mathbb{E}\left[\theta_i\right]\right)^{4-k-k_1}\mathbb{E}\left[\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^k\theta_i^{k_1}\right]+o_p(1) \\
&= \mathbb{E}\left[\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^4\right]+o_p(1).
\end{aligned}$$

□

Lemma C.4. *Under Assumption 3.2,*

$$\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c}-\bar{\hat{\theta}}_c\right)^3\left(u_i-\bar{u}\right)\to_p\mathbb{E}\left[\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^3u_i\right],$$

$$\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c}-\bar{\hat{\theta}}_c\right)^2\left(\theta_i-\bar{\theta}\right)\left(u_i-\bar{u}\right)\to_p\mathbb{E}\left[\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^3u_i\right].$$

Proof. For any $s > 0$ and integers $k_1 \ge 0$, $0 \le k_2 \le 3$, by Chebyshev's inequality,

$$\Pr\left(\left|\frac{1}{n}\sum_{i=1}^{n}c_i^{k_1}\bar{X}_i^{k_2}u_i-\mathbb{E}\left[c_i^{k_1}\bar{X}_i^{k_2}u_i\right]\right|>s\right)\le\frac{\mathbb{E}\left[\bar{X}_i^{2k_2}u_i^2\right]}{ns^2}\to 0. \tag{9}$$

For $0 \le k_3 \le 3$,

$$\mathbb{E}\left[c_i^{k_1}\theta_i^{k_3}u_i\right]=\mathbb{E}\left[c_i^{k_1}\right]\mathbb{E}\left[\theta_i^{k_3}u_i\right]\to\mathbb{E}\left[\theta_i^{k_3}u_i\right]. \qquad \text{(By Lemma B.1 1f)}$$

For $k_4 = 4$, by independence and (8),

$$\begin{aligned}\left|\mathbb{E}\left[c_i^{k_1}\theta_i^{k_3}\bar{\epsilon}_i^{k_4}u_i^2\right]\right| \leq \mathbb{E}\left[\left|\theta_i^{k_3}u_i^2\right|\right]\mathbb{E}\left[\bar{\epsilon}_i^{k_4}\right] &= \mathbb{E}\left[\left|\theta_i^{k_3}u_i^2\right|\right]\mathbb{E}\left[\frac{1}{J_i^3}\mathbb{E}\left[\epsilon_{i,j}^4 \mid J_i, \sigma_i^2\right] + \frac{3\left(J_i - 1\right)}{J_i^3}\sigma_i^2\right] \\ &\leq \mathbb{E}\left[\left|\theta_i^{k_3}u_i^2\right|\right]\mathbb{E}\left[\frac{1}{J_i^3}K\sigma_i^4 + \frac{3\left(J_i-1\right)}{J_i^3}\sigma_i^2\right] \\ &= \mathbb{E}\left[\left|\theta_i^{k_3}u_i^2\right|\right]\left(\mathbb{E}\left[\frac{1}{J_i^3}\right]K\mathbb{E}\left[\sigma_i^4\right] + \mathbb{E}\left[\frac{3\left(J_i-1\right)}{J_i^3}\right]\mathbb{E}\left[\sigma_i^2\right]\right) \to 0.\end{aligned}$$

Then by Jensen's inequality, for $1 \leq k_4 \leq 4$,

$$\left|\mathbb{E}\left[c_i^{k_1}\theta_i^{k_3}\bar{\epsilon}_i^{k_4}u_i^2\right]\right| \to 0.$$

Therefore for $1 \leq k_2 \leq 4$,

$$\begin{aligned}&\mathbb{E}\left[c_i^{k_1}\bar{X}_i^{k_2}u_i\right] \\ &= \sum_{k_4=0}^{k_2}\binom{k_2}{k_4}\mathbb{E}\left[c_i^{k_1}\theta_i^{k_2-k_4}\bar{\epsilon}_i^{k_4}u_i\right] \\ &\to \mathbb{E}\left[\theta_i^{k_2}u_i\right].\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c} - \bar{\hat{\theta}}_c\right)^3\left(u_i - \bar{u}\right) \\ &= \frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c} - \bar{\hat{\theta}}_c\right)^3 u_i + o_p(1) \\ &= \mathbb{E}\left[\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)^3 u_i\right] + o_p(1).\end{aligned}$$

The second result is shown by combining the proof procedures of Lemma C.3 and the above. □

Lemma C.5. *Under Assumption 3.2,*

$$\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c} - \bar{\hat{\theta}}_c\right)^2\left(u_i - \bar{u}\right)^2 \to_p \mathbb{E}\left[u_i^2\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)^2\right]$$

Proof. For any $s > 0$ and integers $k_1 \geq 0$, $0 \leq k_2 \leq 3$, by Chebyshev's inequality,

$$\Pr\left(\left|\frac{1}{n}\sum_{i=1}^{n} c_i^{k_1}\bar{X}_i^{k_2}u_i^2 - \mathbb{E}\left[c_i^{k_1}\bar{X}_i^{k_2}u_i^2\right]\right| > s\right) \leq \frac{\mathbb{E}\left[\bar{X}_i^{2k_2}u_i^4\right]}{ns^2} \to 0.$$

For $0 \leq k_3 \leq 3$,

$$\mathbb{E}\left[c_i^{k_1}\theta_i^{k_3}u_i^2\right] = \mathbb{E}\left[c_i^{k_1}\right]\mathbb{E}\left[\theta_i^{k_3}u_i^2\right] \to \mathbb{E}\left[\theta_i^{k_3}u_i^2\right]. \qquad \text{(By Lemma B.1 1f)}$$

For $k_4 = 4$, by independence and (8),

$$\begin{aligned}\left|\mathbb{E}\left[c_i^{k_1}\theta_i^{k_3}\bar{\epsilon}_i^{k_4}u_i^2\right]\right| \leq \mathbb{E}\left[\left|\theta_i^{k_3}u_i^2\right|\right]\mathbb{E}\left[\bar{\epsilon}_i^{k_4}\right] &= \mathbb{E}\left[\left|\theta_i^{k_3}u_i^2\right|\right]\mathbb{E}\left[\frac{1}{J_i^3}\mathbb{E}\left[\epsilon_{i,j}^4 \mid J_i, \sigma_i^2\right] + \frac{3(J_i-1)}{J_i^3}\sigma_i^2\right] \\ &\leq \mathbb{E}\left[\left|\theta_i^{k_3}u_i^2\right|\right]\mathbb{E}\left[\frac{1}{J_i^3}K\sigma_i^4 + \frac{3(J_i-1)}{J_i^3}\sigma_i^2\right] \\ &= \mathbb{E}\left[\left|\theta_i^{k_3}u_i^2\right|\right]\left(\mathbb{E}\left[\frac{1}{J_i^3}\right]K\mathbb{E}\left[\sigma_i^4\right] + \mathbb{E}\left[\frac{3(J_i-1)}{J_i^3}\right]\mathbb{E}\left[\sigma_i^2\right]\right) \to 0.\end{aligned}$$

Then by Jensen's inequality, for $1 \leq k_4 \leq 4$,

$$\left|\mathbb{E}\left[c_i^{k_1}\theta_i^{k_3}\bar{\epsilon}_i^{k_4}u_i^2\right]\right| \to 0.$$

Therefore for $1 \leq k_2 \leq 4$,

$$\begin{aligned}&\mathbb{E}\left[c_i^{k_1}\bar{X}_i^{k_2}u_i^2\right] \\ &= \sum_{k_4=0}^{k_2}\binom{k_2}{k_4}\mathbb{E}\left[c_i^{k_1}\theta_i^{k_2-k_4}\bar{\epsilon}_i^{k_4}u_i^2\right] \\ &\to \mathbb{E}\left[\theta_i^{k_2}u_i^2\right].\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c} - \bar{\hat{\theta}}_c\right)^2(u_i - \bar{u})^2 \\ &= \frac{1}{n}\sum_{i=1}^{n}\left(\hat{\theta}_{i,c} - \bar{\hat{\theta}}_c\right)^2 u_i^2 + o_p(1) \\ &= \mathbb{E}\left[(\theta_i - \mathbb{E}[\theta_i])^2 u_i^2\right] + o_p(1).\end{aligned}$$

□

D Lemmas and Proofs for Correlated J_i and σ_i^2

Firstly, for simplicity we abbreviate notations and denote $\hat{\beta}_{c,\mathrm{HE}}$ as $\hat{\beta}_c$, $\hat{\beta}_{\mathrm{HE}}$ as $\hat{\beta}$, $\hat{\theta}_{i,c,\mathrm{HE}}$ as $\hat{\theta}_{i,c}$, $\hat{\theta}_{i,\mathrm{HE}}$ as $\hat{\theta}_i$ and $\mathrm{Var}(\theta_i)$ as V. We keep the subscripts for those related to HO.

We next define the shrinkage weight

$$c_{i,\mathrm{HO}} := \frac{V}{\frac{\mathbb{E}[\sigma_i^2]}{J_i} + V},$$

where $V = \mathrm{Var}(\theta_i)$. And without affecting the results, we define

$$\hat{\theta}_{i,\mathrm{HO}} := \frac{\hat{\sigma}_\theta^2}{\frac{1}{J_i}\hat{\sigma}^2 + \hat{\sigma}_\theta^2}\bar{X}_i + \frac{\frac{1}{J_i}\hat{\sigma}^2}{\frac{1}{J_i}\hat{\sigma}^2 + \hat{\sigma}_\theta^2}\bar{X}.$$

Assumption D.1. *1. J_i is independent of θ_i and u_i. $J_i \perp\!\!\!\perp \epsilon_{i,j} \mid \sigma_i^2$. $J_i \geq 3, a.s.$*

2. $\mathbb{E}[u_i] = 0$, $\mathbb{E}(u_i\theta_i) = 0$. Y_i has finite fourth moments.

3. $\mathbb{E}[\epsilon_{i,j} \mid \sigma_i^2] = 0$, $\mathbb{E}\left[\epsilon_{i,j}^2 \mid \sigma_i^2\right] = \sigma_i^2$, $\mathbb{E}\left[|\epsilon_{i,j}|^L \mid \sigma_i^2\right] \leq K\sigma_i^L, L \geq 3$. Also, $\epsilon_{i,j} \perp\!\!\!\perp \theta_i$.

4. $u_i \perp\!\!\!\perp \epsilon_{i,j} \big| \theta_i$.

5. θ_i has finite fourth moments.

6. σ_i^2 has finite sixteenth moments.

Assumption D.2.

$$n^{\frac{3}{2}}\mathbb{E}\left[\frac{1}{J_i^3}\right] \to \kappa^3$$

Remark D.1. Since we assume $J_i \geq 3$, Assumption D.2 also implies that

$$n\mathbb{E}\left[\frac{1}{J_i^2}\right] \leq \left(n^{\frac{3}{2}}\mathbb{E}\left[\frac{1}{J_i^3}\right]\right)^{2/3} = O(1). \qquad \text{(Jensen's inequality)}$$

Lemma D.1. *Under Assumption D.1, Assumption D.2, we have:*

1. *Properties of* $\bar{\epsilon}_i$, $\bar{X}_i$, $\hat{\sigma}_i^2$, c_i, $c_{i,HO}$:

   For any integer $k_1 \in \{1,2\}$, $k_2 \geq 1$, *we have*

   (a) $\bar{\epsilon}_i = O_p(n^{-1/4})$, *and* $\mathbb{E}\left[|\bar{\epsilon}_i|^{k_1}\right] \to 0$.

   (b) $\bar{X}_i = \theta_i + O_p(n^{-1/4})$, *and* $\mathbb{E}\left[\left|\bar{X}_i\right|^{k_1}\right] \to \mathbb{E}\left[|\theta_i|^{k_1}\right]$.

   (c) $\hat{\sigma}_i^2 = \sigma_i^2 + O_p(n^{-1/4})$, *and* $\mathbb{E}\left[|\hat{\sigma}_i^2 - \sigma_i^2|^{k_1}\right] \to 0$.

   (d) $\mathbb{E}\left[\frac{1}{J_i}\hat{\sigma}_i^2\right] \to 0$, *and* $\mathbb{E}\left[\left(\frac{1}{J_i}\hat{\sigma}_i^2\right)^2\right] \to 0$.

   (e) $\mathbb{E}\left[\left(\frac{1}{J_i}\hat{\sigma}_i^2\right)^4\right] = o(n^{-1/2})$.

   (f) $c_i^{k_2} = 1 + O_p(n^{-1/2})$, *and* $\mathbb{E}\left[c_i^{k_2}\right] = 1 + O(n^{-1/2})$.

   (g) $c_{i,\mathrm{HO}}^{k_2} = 1 + O_p(n^{-1/2})$, *and* $\mathbb{E}\left[c_{i,\mathrm{HO}}^{k_2}\right] = 1 + O(n^{-1/2})$.

2. *Properties of sample moments of* $\bar{\epsilon}_i$, c_i, θ_i, $c_{i,\mathrm{HO}}$:

   For any integer $k_1 \geq 0$[2], $0 \leq k_2 \leq 2$, *and* $k_3, k_4 \in \{0,1\}$, *we have*

$$\frac{1}{n}\sum_{i=1}^{n} c_i^{k_1}\left(\theta_i - k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2} = \mathbb{E}\left[\left(\theta_i - k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2}\right] + O_p(n^{-1/2}),$$

$$\frac{1}{n}\sum_{i=1}^{n} c_i^{k_1}\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)^{k_4}\bar{\epsilon}_i = o_p(n^{-1/2}),$$

$$\frac{1}{n}\sum_{i=1}^{n} c_i^{k_1}\left(\theta_i - k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2}\bar{\epsilon}_i^2 = O_p(n^{-1/2}).$$

   And similar results hold for $c_{i,HO}$.

3. *Properties of sample means of* θ_i, $\bar{X}_i$, $\hat{\sigma}_i^2$:

   (a) $\bar{\theta} = \mathbb{E}\left[\theta_i\right] + O_p(n^{-1/2})$.

   (b) $\bar{X} = \mathbb{E}\left[\theta_i\right] + O_p(n^{-1/2})$.

   (c) $\frac{1}{n}\sum_{i=1}^{n}\frac{1}{J_i}\hat{\sigma}_i^2 = O_p(n^{-1/2})$

4. *Properties of sample moments of* ϵ_i, c_i, u_i,$c_{i,HO}$:

   For any integer $k \geq 0$, *we have*

[2]For statements like this, if $k = 0$ is on the exponent, it means that the term is 1.

(a) $\frac{1}{n}\sum_{i=1}^{n} c_i^k u_i = O_p(n^{-1/2})$.

(b) $\frac{1}{n}\sum_{i=1}^{n} c_i^k u_i \bar{\epsilon}_i = o_p(n^{-1/2})$.

And similar results hold for $c_{i,\mathrm{HO}}$.

Proof of Lemma D.1. Below we take any $s > 0$,

1. Properties of $\bar{\epsilon}_i$, $\bar{X}_i$, $\hat{\sigma}_i^2$, c_i:

   (a) By Markov's inequality,

   $$\Pr\left(n^{1/4}\left|\bar{\epsilon}_i\right| > s\right) \leq \frac{\sqrt{n}\mathbb{E}\left[\bar{\epsilon}_i^2\right]}{s^2} = \frac{\sqrt{n}\mathbb{E}\left[\frac{1}{J_i}\sigma_i^2\right]}{s^2} \leq \frac{\sqrt{n\mathbb{E}\left[\frac{1}{J_i^2}\right]\mathbb{E}\left[\sigma_i^4\right]}}{s^2}.$$

   Since $n\mathbb{E}\left[\frac{1}{J_i^2}\right] \to \kappa^2$, we have $\bar{\epsilon}_i = O_p(n^{-1/4})$.
   Since

   $$\mathbb{E}\left[\bar{\epsilon}_i^2\right] = \mathbb{E}\left[\frac{1}{J_i}\sigma_i^2\right] \leq \sqrt{\mathbb{E}\left[\frac{1}{J_i^2}\right]\mathbb{E}\left[\sigma_i^4\right]} \to 0,$$

   combined with the Cauchy-Schwarz inequality, we have the rest of the results.

   (b) This follows from 1a.

   (c) By Markov's inequality and (8),

   $$\Pr\left(n^{1/4}\left|\hat{\sigma}_i^2 - \sigma_i^2\right| > s\right) \leq \frac{\sqrt{n}\mathbb{E}\left[\left(\hat{\sigma}_i^2 - \sigma_i^2\right)^2\right]}{s^2} \leq \frac{2\sqrt{n}\mathbb{E}\left[\frac{K}{J_i}\sigma_i^4\right]}{s^2} \leq \frac{2K\sqrt{n\mathbb{E}\left[\frac{1}{(J_i)^2}\right]\mathbb{E}\left[\sigma_i^8\right]}}{s^2}.$$

   Due to Assumption D.2, we have $\hat{\sigma}_i^2 = \sigma_i^2 + O_p(n^{-1/4})$.
   Also,

   $$\mathbb{E}\left[\left(\hat{\sigma}_i^2 - \sigma_i^2\right)^2\right] \leq 2\mathbb{E}\left[\frac{K}{J_i}\sigma_i^4\right] \leq 2K\sqrt{\mathbb{E}\left[\frac{1}{(J_i)^2}\right]\mathbb{E}\left[\sigma_i^8\right]} \to 0,$$

   combined with the Cauchy-Schwarz inequality, we have the rest of the results.

(d) We have

$$0 \leq \mathbb{E}\left[\frac{1}{J_i}\hat{\sigma}_i^2\right] = \mathbb{E}\left[\frac{1}{J_i}\sigma_i^2\right] \leq \sqrt{\mathbb{E}\left[\frac{1}{J_i^2}\right]\mathbb{E}\left[\sigma_i^4\right]} \to 0,$$

$$\mathbb{E}\left[\left(\frac{1}{J_i}\hat{\sigma}_i^2\right)^2\right] = \mathbb{E}\left[\frac{1}{J_i^2}\left(\frac{K\sigma_i^4}{J_i} + \sigma_i^4\right)\right] \to 0.$$

(e) From Theorem 2 of Angelova (2012), the fourth moment of $\hat{\sigma}_i^2$ is

$$\begin{aligned}
\mathbb{E}\left[\left(\hat{\sigma}_i^2\right)^4 \mid J_i, \sigma_i^2\right] = \mu_2^4 &+ \frac{6\mu_2^2(\mu_4 - \mu_2^2)}{J_i} + \frac{4\mu_6\mu_2 + 3\mu_4^2 - 18\mu_4\mu_2^2 - 24\mu_3^2\mu_2 + 23\mu_2^4}{J_i^2} \\
&+ \frac{\mu_8 - 4\mu_6\mu_2 - 24\mu_5\mu_3 - 3\mu_4^2 + 72\mu_4\mu_2^2 + 96\mu_3^2\mu_2 - 86\mu_2^4}{J_i^3} \\
&+ \frac{4(6\mu_6\mu_2 + 6\mu_4^2 - 39\mu_4\mu_2^2 - 40\mu_3^2\mu_2 + 45\mu_2^4)}{J_i^3(J_i - 1)} \\
&+ \frac{4(36\mu_4\mu_2^2 - 8\mu_5\mu_3 + 52\mu_3^2\mu_2 - 61\mu_2^4)}{J_i^3(J_i - 1)^2} \\
&+ \frac{8(\mu_4^2 - 6\mu_4\mu_2^2 - 12\mu_3^2\mu_2 + 15\mu_2^4)}{J_i^3(J_i - 1)^3},
\end{aligned}$$

where $\mu_k, k = 1, ..., 8$ are the k-th raw moments of $\epsilon_{i,j} \mid \sigma_i^2$. By Assumption D.1.3, we have $\mu_k \leq K\sigma_i^k$ for some constant K. By the law of iterated expectations and Assumption D.2, for the first term we have

$$\sqrt{n}\mathbb{E}\left[\frac{1}{J_i^4}\sigma_i^8\right] \leq \sqrt{n\mathbb{E}\left[\frac{1}{J_i^8}\right]\mathbb{E}\left[\sigma_i^{16}\right]} \leq \sqrt{n\mathbb{E}\left[\frac{1}{J_i^3}\right]\mathbb{E}\left[\sigma_i^{16}\right]} \to 0.$$

For the rest of the terms, we have the same result. Therefore,

$$\sqrt{n}\mathbb{E}\left[\left(\frac{1}{J_i}\hat{\sigma}_i^2\right)^4\right] = o(n^{-1/2}).$$

(f) By the mean value theorem,

$$
\begin{aligned}
c_i^{k_2} &= \left(\frac{V}{V + \frac{1}{J_i}\hat{\sigma}_i^2}\right)^{k_2} \\
&= 1 - k_2 \frac{1}{V J_i}\hat{\sigma}_i^2 + \frac{k_2 (k_2+1)}{2} \frac{1}{(1+\omega_i)^{k_2+2}} \frac{1}{V^2 J_i^2} \left(\hat{\sigma}_i^2\right)^2,
\end{aligned}
$$

where ω_i is between 0 and $\frac{1}{VJ_i}\hat{\sigma}_i^2$. Therefore,

$$
\begin{aligned}
\left|\sqrt{n}\mathbb{E}\left[1 - c_i^{k_2}\right]\right| &\le \frac{k_2}{V}\sqrt{n}\mathbb{E}\left[\frac{\sigma_i^2}{J_i}\right] \\
&\quad + \frac{k_2(k_2+1)}{2V^2}\sqrt{n}\mathbb{E}\left[\frac{1}{J_i^2}\left(\sigma_i^4 + \frac{K}{J_i}\sigma_i^4\right)\right] && \text{(By (8))} \\
&\le \frac{k_2}{V}\sqrt{\mathbb{E}\left[\sigma_i^4\right] n\mathbb{E}\left[\frac{1}{J_i^2}\right]} \\
&\quad + \frac{k_2(k_2+1)}{2V^2}\sqrt{\mathbb{E}\left[\sigma_i^8\right] n\mathbb{E}\left[\frac{1}{J_i^4}\right]} \\
&\quad + \frac{k_2(k_2+1)K}{2V^2}\sqrt{\mathbb{E}\left[\sigma_i^8\right] n\mathbb{E}\left[\frac{1}{J_i^6}\right]} && \text{(Cauchy-Schwarz)} \\
&= O(1) + o(1) + o(1) && \text{(By Assumption D.2)}
\end{aligned}
$$

Then we have the second result. By Markov's inequality, we have the first result.

(g) The proof is similar to 1f. By the mean value theorem,

$$
\begin{aligned}
c_{i,\mathrm{HO}}^{k_2} &= \left(\frac{V}{V + \frac{\mathbb{E}\left[\sigma_i^2\right]}{J_i}}\right)^{k_2} \\
&= 1 - k_2 \frac{\mathbb{E}\left[\sigma_i^2\right]}{V J_i} + \frac{k_2(k_2+1)}{2}\frac{1}{(1+\omega_i)^{k_2+2}}\frac{\mathbb{E}\left[\sigma_i^2\right]^2}{V^2 J_i^2}.
\end{aligned}
$$

It is easy to see that the rest of the proof is similar to 1f.

2. by Chebyshev's inequality (k_2 can be replaced by k_4),

$$\Pr\left(\sqrt{n}\left|\frac{1}{n}\sum_{i=1}^{n}c_i^{k_1}\left(\theta_i-k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2}-\mathbb{E}\left[c_i^{k_1}\left(\theta_i-k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2}\right]\right|>s\right)$$
$$\leq\frac{\mathbb{E}\left[c_i^{2k_1}\left(\theta_i-k_3\mathbb{E}\left[\theta_i\right]\right)^{2k_2}\right]}{s^2}\leq\frac{\mathbb{E}\left[\left(\theta_i-k_3\mathbb{E}\left[\theta_i\right]\right)^{2k_2}\right]}{s^2}$$

$$\Pr\left(\sqrt{n}\left|\frac{1}{n}\sum_{i=1}^{n}c_i^{k_1}\left(\theta_i-k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2}\bar{\epsilon}_i-\mathbb{E}\left[c_i^{k_1}\left(\theta_i-k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2}\bar{\epsilon}_i\right]\right|>s\right)$$
$$\leq\frac{\mathbb{E}\left[c_i^{2k_1}\left(\theta_i-k_3\mathbb{E}\left[\theta_i\right]\right)^{2k_2}\bar{\epsilon}_i^2\right]}{s^2}\leq\frac{\mathbb{E}\left[\left(\theta_i-k_3\mathbb{E}\left[\theta_i\right]\right)^{2k_2}\right]\sqrt{\mathbb{E}\left[\frac{1}{J_i^2}\right]\mathbb{E}\left[\sigma_i^4\right]}}{s^2}\to 0,$$

$$\Pr\left(\sqrt{n}\left|\frac{1}{n}\sum_{i=1}^{n}c_i^{k_1}\left(\theta_i-k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2}\bar{\epsilon}_i^2-\mathbb{E}\left[c_i^{k_1}\left(\theta_i-k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2}\bar{\epsilon}_i^2\right]\right|>s\right)$$
$$\leq\frac{\mathbb{E}\left[c_i^{2k_1}\left(\theta_i-k_3\mathbb{E}\left[\theta_i\right]\right)^{2k_2}\bar{\epsilon}_i^4\right]}{s^2}$$
$$\leq\frac{\mathbb{E}\left[\left(\theta_i-k_3\mathbb{E}\left[\theta_i\right]\right)^{2k_2}\right]\left(\sqrt{\mathbb{E}\left[\frac{1}{J_i^6}\right]\mathbb{E}\left[K^2\sigma_i^8\right]}+\sqrt{\mathbb{E}\left[\frac{(J_i-1)^2}{J_i^6}\right]\mathbb{E}\left[\sigma_i^8\right]}\right)}{s^2}\to 0.$$

Meanwhile,

$$\sqrt{n}\mathbb{E}\left[c_i^{k_1}\left(\theta_i-k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2}\right]=\sqrt{n}\mathbb{E}\left[c_i^{k_1}\right]\mathbb{E}\left[\left(\theta_i-k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2}\right]$$
$$=\sqrt{n}\mathbb{E}\left[\left(\theta_i-k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2}\right]+O(1),\qquad\text{(By 1f)}$$

by the mean value theorem ($k_4 = 0$) and (8),

$$
\begin{aligned}
\left|\sqrt{n}\mathbb{E}\left[c_i^{k_1}\bar{\epsilon}_i\right]\right| &\leq 0 + \frac{k_1}{V}\sqrt{n}\mathbb{E}\left[\frac{1}{J_i}\hat{\sigma}_i^2\bar{\epsilon}_i\right] \\
&\quad + \frac{k_1(k_1+1)}{2V^2}\sqrt{n}\mathbb{E}\left[\frac{1}{J_i^2}\left(\hat{\sigma}_i^2\right)^2\bar{\epsilon}_i\right] \\
&\leq \frac{k_1}{V}\sqrt{\sqrt{n}\mathbb{E}\left[\frac{1}{J_i^2}\left(\sigma_i^4 + \frac{K}{J_i}\sigma_i^4\right)\right]\sqrt{n}\mathbb{E}\left[\frac{1}{J_i}\sigma_i^2\right]} \\
&\quad + \frac{k_1(k_1+1)}{2V^2}\sqrt{\sqrt{n}\mathbb{E}\left[\frac{1}{J_i^4}\left(\hat{\sigma}_i^2\right)^4\right]\sqrt{n}\mathbb{E}\left[\frac{1}{J_i}\sigma_i^2\right]} \qquad \text{(Cauchy-Schwarz)} \\
&\leq \frac{k_1}{V}\sqrt{\sqrt{n\mathbb{E}\left[\frac{1}{J_i^4}\right]\mathbb{E}\left[\sigma_i^8\right] + n\mathbb{E}\left[\frac{1}{J_i^6}\right]K^2\mathbb{E}\left[\sigma_i^8\right]}\sqrt{n\mathbb{E}\left[\frac{1}{J_i^2}\right]\mathbb{E}\left[\sigma_i^4\right]}} \\
&\quad + \frac{k_1(k_1+1)}{2V^2}\sqrt{\sqrt{n}\mathbb{E}\left[\frac{1}{J_i^4}\left(\hat{\sigma}_i^2\right)^4\right]\sqrt{n\mathbb{E}\left[\frac{1}{J_i^2}\right]\mathbb{E}\left[\sigma_i^4\right]}} \\
&= o(1)\cdot O(1) + o(1)\cdot O(1)
\end{aligned}
$$

where the last step follows from Assumption D.2 and 1e,

by the independence in Assumption 3.1.3 ($k_4 = 1$),

$$
\mathbb{E}\left[c_i^{k_1}\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)\bar{\epsilon}_i\right] = \mathbb{E}\left[\mathbb{E}\left(c_i^{k_1}\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)\bar{\epsilon}_i\middle|\epsilon\right)\right] = 0,
$$

and also by Assumption 3.1.3,

$$
\left|\sqrt{n}\mathbb{E}\left[c_i^{k_1}\left(\theta_i - k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2}\bar{\epsilon}_i^2\right]\right| \leq \mathbb{E}\left[\left|\left(\theta_i - k_3\mathbb{E}\left[\theta_i\right]\right)^{k_2}\right|\right]\sqrt{n\mathbb{E}\left[\frac{1}{J_i^2}\right]\mathbb{E}\left[\sigma_i^4\right]},
$$

where $n\mathbb{E}\left[\frac{1}{J_i^2}\right] = O(1)$. Therefore we have the results. For $c_{i,\text{HO}}$, the proof is similar.

3. (a) Follows from the finite fourth moment and the inequality below for any $s > 0$,

$$
\Pr\left(\left|\sqrt{n}\left(\bar{\theta} - \mathbb{E}\left[\theta_i\right]\right)\right| > s\right) \leq \frac{\mathbb{E}\left[\theta_i^2\right]}{s^2}. \qquad \text{(Chebyshev's inequality)}
$$

(b) Follows from the property of $\bar{\theta}$ and the second property of 2:

$$\bar{X} = \bar{\theta} + \bar{\epsilon} = \mathbb{E}\left[\theta_i\right] + O_p(n^{-1/2}) + o_p(n^{-1/2}) = \mathbb{E}\left[\theta_i\right] + O_p(n^{-1/2}).$$

(c) By Chebyshev's inequality and (8),

$$\Pr\left(\sqrt{n}\left|\frac{1}{n}\sum_{i=1}^{n}\frac{1}{J_i}\hat{\sigma}_i^2 - \mathbb{E}\left[\frac{1}{J_i}\hat{\sigma}_i^2\right]\right| > s\right) \leq \frac{\mathbb{E}\left[\frac{1}{J_i^2}\left(\frac{K\sigma_i^4}{J_i} + \sigma_i^4\right)\right]}{s^2} \to 0,$$

and also

$$\sqrt{n}\mathbb{E}\left[\frac{1}{J_i}\hat{\sigma}_i^2\right] \leq \sqrt{n\mathbb{E}\left[\frac{1}{J_i^2}\right]\mathbb{E}\left[\sigma_i^4\right]} = O(1).$$

Thus

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{J_i}\hat{\sigma}_i^2 = O_p(n^{-1/2}).$$

4. (a) Since $u_i \perp\!\!\!\perp \epsilon_{i,j} \mid \theta_i$, by Chebyshev's inequality,

$$\Pr\left(\sqrt{n}\left|\frac{1}{n}\sum_{i=1}^{n}c_i^k u_i\right| > s\right) \leq \frac{\mathbb{E}\left[c_i^{2k}u_i^2\right]}{s^2} \leq \frac{\sigma_u^2}{s^2}$$

Thus

$$\frac{1}{n}\sum_{i=1}^{n}c_i^k u_i = O_p(n^{-1/2}).$$

(b) By Chebyshev's inequality,

$$\Pr\left(\sqrt{n}\left|\frac{1}{n}\sum_{i=1}^{n}c_i^k u_i\bar{\epsilon}_i\right| > s\right) \leq \frac{\mathbb{E}\left[c_i^{2k}u_i^2\bar{\epsilon}_i^2\right]}{s^2} \leq \frac{\sigma_u^2}{s^2}\sqrt{\mathbb{E}\left[\frac{1}{J_i^4}\right]\mathbb{E}\left[\sigma_i^8\right]} \to 0.$$

Thus

$$\frac{1}{n}\sum_{i=1}^{n}c_i^k u_i\bar{\epsilon}_i = o_p(n^{-1/2}).$$

For $c_{i,\mathrm{HO}}$, the proof is similar.

□

Lemma D.2. *Under Assumption D.1, Assumption D.2, we have*

$$\frac{1}{n}\sum_{i=1}^{n} c_i^2 \bar{\epsilon}_i^2 = \frac{1}{n}\sum_{i=1}^{n} c_i^2 \frac{1}{J_i}\hat{\sigma}_i^2 + o_p(n^{-1/2}).$$

For $c_{i,\mathrm{HO}}$, the result is similar.

Proof of Lemma D.2. First, notice that the difference

$$\begin{aligned}
\bar{\epsilon}_i^2 - \frac{1}{J_i}\hat{\sigma}_i^2 &= \bar{\epsilon}_i^2 - \frac{1}{J_i\left(J_i-1\right)}\sum_{j=1}^{J_i}\epsilon_{i,j}^2 + \frac{1}{J_i-1}\bar{\epsilon}_i^2 \\
&= \frac{1}{J_i\left(J_i-1\right)}\left(\sum_{j=1}^{J_i}\epsilon_{i,j}\right)^2 - \frac{1}{J_i\left(J_i-1\right)}\sum_{j=1}^{J_i}\epsilon_{i,j}^2 \\
&= \frac{2}{J_i\left(J_i-1\right)}\sum_{k\leq j}\epsilon_{i,k}\epsilon_{i,j}.
\end{aligned}$$

Then the properties of the difference are

$$\begin{aligned}
&\mathbb{E}\left[\bar{\epsilon}_i^2 - \frac{1}{J_i}\hat{\sigma}_i^2\right] = 0, \\
&\mathbb{E}\left[\left(\bar{\epsilon}_i^2 - \frac{1}{J_i}\hat{\sigma}_i^2\right)^2 \mid J_i, \sigma_i^2\right] = \frac{2\sigma_i^4}{J_i\left(J_i-1\right)}.
\end{aligned}$$

For c_i^2, by the mean value theorem,

$$\begin{aligned}
c_i^2 &= \left(\frac{V}{V + \frac{1}{J_i}\hat{\sigma}_i^2}\right)^2 \\
&= 1 - 2\frac{1}{VJ_i}\hat{\sigma}_i^2 + 3\frac{1}{\left(1+\omega_i\right)^4}\frac{1}{V^2J_i^2}\left(\hat{\sigma}_i^2\right)^2,
\end{aligned}$$

where ω_i is between 0 and $\frac{1}{VJ_i}\hat{\sigma}_i^2$. Therefore,

$$
\begin{aligned}
&\left|\sqrt{n}\mathbb{E}\left[c_i^2\left(\bar{\epsilon}_i^2-\frac{1}{J_i}\hat{\sigma}_i^2\right)\right]\right| \\
&\leq \left|\sqrt{n}\mathbb{E}\left[\left(\bar{\epsilon}_i^2-\frac{1}{J_i}\hat{\sigma}_i^2\right)\right]\right| + \frac{2}{V}\left|\sqrt{n}\mathbb{E}\left[\frac{1}{J_i}\hat{\sigma}_i^2\left(\bar{\epsilon}_i^2-\frac{1}{J_i}\hat{\sigma}_i^2\right)\right]\right| \\
&\quad + \frac{3}{V^2}\left|\sqrt{n}\mathbb{E}\left[\frac{1}{(1+\omega_i)^4}\frac{1}{J_i^2}\left(\hat{\sigma}_i^2\right)^2\left(\bar{\epsilon}_i^2-\frac{1}{J_i}\hat{\sigma}_i^2\right)\right]\right| \\
&\leq 0 + \frac{2}{V}\sqrt{n\mathbb{E}\left[\frac{1}{J_i^2}\left(\hat{\sigma}_i^2\right)^2\right]\mathbb{E}\left[\left(\bar{\epsilon}_i^2-\frac{1}{J_i}\hat{\sigma}_i^2\right)^2\right]} \\
&\quad + \frac{3}{V^2}\sqrt{n\mathbb{E}\left[\frac{1}{J_i^4}\left(\hat{\sigma}_i^2\right)^4\right]\mathbb{E}\left[\left(\bar{\epsilon}_i^2-\frac{1}{J_i}\hat{\sigma}_i^2\right)^2\right]} && \text{(Cauchy-Schwarz)} \\
&\leq \frac{2}{V}\sqrt{\sqrt{n}\mathbb{E}\left[\frac{1}{J_i^2}\left(\frac{K}{J_i}\sigma_i^4+\sigma_i^4\right)\right]\sqrt{n}\mathbb{E}\left[\frac{2\sigma_i^4}{J_i\left(J_i-1\right)}\right]} \\
&\quad + \frac{3}{V^2}\sqrt{\sqrt{n}\mathbb{E}\left[\frac{1}{J_i^4}\left(\hat{\sigma}_i^2\right)^4\right]\sqrt{n}\mathbb{E}\left[\frac{2\sigma_i^4}{J_i\left(J_i-1\right)}\right]}. \\
&\leq \frac{2}{V}\sqrt{\sqrt{n\mathbb{E}\left[\frac{1}{J_i^6}\right]\mathbb{E}\left[K^2\sigma_i^8\right]+n\mathbb{E}\left[\frac{1}{J_i^4}\right]\mathbb{E}\left[\sigma_i^8\right]}\sqrt{n\mathbb{E}\left[\frac{1}{J_i^2\left(J_i-1\right)^2}\right]\mathbb{E}\left[4\sigma_i^8\right]}} \\
&\quad + \frac{3}{V^2}\sqrt{\sqrt{n\mathbb{E}\left[\frac{1}{J_i^4}\left(\hat{\sigma}_i^2\right)^4\right]}\sqrt{n\mathbb{E}\left[\frac{1}{J_i^2\left(J_i-1\right)^2}\right]\mathbb{E}\left[4\sigma_i^8\right]}}. && \text{(Cauchy-Schwarz)}
\end{aligned}
$$

Since we have Assumption D.2 and that

$$\sqrt{n}\mathbb{E}\left[\frac{1}{J_i^4}\left(\hat{\sigma}_i^2\right)^4\right] \to 0,$$

from Lemma D.1.1e, the above converges to 0. Thus, the expectation

$$\sqrt{n}\mathbb{E}\left[c_i^2\left(\bar{\epsilon}_i^2-\frac{1}{J_i}\hat{\sigma}_i^2\right)\right] \to 0.$$

Next, we show the convergence to expectations by Chebyshev's inequality:

$$\Pr\left(\sqrt{n}\left|\frac{1}{n}\sum_{i=1}^{n}c_i^2\left(\bar{\epsilon}_i^2-\frac{1}{J_i}\hat{\sigma}_i^2\right)-\mathbb{E}\left[c_i^2\left(\bar{\epsilon}_i^2-\frac{1}{J_i}\hat{\sigma}_i^2\right)\right]\right|>s\right)$$
$$\leq\frac{\mathbb{E}\left[c_i^4\left(\bar{\epsilon}_i^2-\frac{1}{J_i}\hat{\sigma}_i^2\right)^2\right]}{s^2}\leq\frac{\mathbb{E}\left[\left(\bar{\epsilon}_i^2-\frac{1}{J_i}\hat{\sigma}_i^2\right)^2\right]}{s^2}=\frac{\mathbb{E}\left[\frac{2\sigma_i^4}{J_i(J_i-1)}\right]}{s^2}\to 0.$$

And the proof for c_i is complete.

For $c_{i,\text{HO}}$, the proof is similar. □

Lemma D.3. *Under Assumption D.1, Assumption D.2, we have*

$$\hat{V}=V+O_p(n^{-1/2}).$$

Proof of Lemma D.3. By Lemma D.1,

$$\begin{aligned}\hat{V}&=\frac{1}{n}\sum_{i=1}^{n}\left(\bar{X}_i-\bar{X}\right)^2-\frac{n-1}{n^2}\sum_{i=1}^{n}\frac{1}{J_i}\hat{\sigma}_i^2\\&=\frac{1}{n}\sum_{i=1}^{n}\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^2+\frac{1}{n}\sum_{i=1}^{n}\bar{\epsilon}_i^2-\left(\bar{\theta}-\mathbb{E}\left[\theta_i\right]\right)^2-\bar{\epsilon}^2+\frac{2}{n}\sum_{i=1}^{n}\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)\bar{\epsilon}_i\\&\quad-2\left(\bar{\theta}-\mathbb{E}\left[\theta_i\right]\right)\bar{\epsilon}-\frac{n-1}{n^2}\sum_{i=1}^{n}\frac{1}{J_i}\hat{\sigma}_i^2\\&=V+O_p(n^{-1/2})+O_p(n^{-1/2})-O_p(n^{-1})-o_p(n^{-1})+o_p(n^{-1/2})\\&\quad-o_p(n^{-1})-O_p(n^{-1/2})\\&=V+O_p(n^{-1/2}).\end{aligned}$$

□

Lemma D.4. *Under Assumption D.1, Assumption D.2, we have*

$$\hat{V}_{HO}=V+O_p(n^{-1/2}).$$

Proof of Lemma D.4. By conventional law of large number and central limit theorem arguments,

$$\begin{aligned}\hat{V}_{\text{HO}} &= \frac{1}{n}\sum_{i=1}^{n}\left(\bar{X}_{1,i}-\bar{X}_1\right)\left(\bar{X}_{2,i}-\bar{X}_2\right)\\ &= V + O_p(n^{-1/2})\end{aligned}$$

□

Proposition D.1 (Proposition 3.4)**.** *Suppose the asymptotic framework satisfies Assumption D.2. Then under Assumption D.1 we have*

$$\sqrt{n}\left(\hat{\beta}-\beta\right)\to_d N\left(0,\frac{\mathbb{E}\left[u_i^2\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^2\right]}{V^2}\right).$$

Proof of Proposition D.1. Based on a similar set of lemmas to Appendix B, the claim follows from the same arguments as in Theorem 3.3. □

Proposition D.2 (Proposition 3.2)**.** *Suppose the asymptotic framework satisfies Assumption D.2. Then under Assumption D.1, there exist cases where*

$$\sqrt{n}\left(\hat{\beta}_{HO}-\beta\right)\to_d N\left(\frac{\beta}{\text{Var}\left(\theta_i\right)},\frac{\mathbb{E}\left[u_i^2\left(\theta_i-\mathbb{E}\left[\theta_i\right]\right)^2\right]}{\left(\text{Var}\left(\theta_i\right)\right)^2}\right).$$

Proof of Proposition D.2. For the shrinkage weight

$$c_{i,\text{HO}} = \frac{V}{\frac{1}{J_i}\mathbb{E}\left[\sigma_i^2\right]+V},$$

we first show properties of the regression coefficients $\hat{\beta}_{\text{HO},c}$, and then show that $\sqrt{n}\left(\hat{\beta}_{\text{HO}}-\beta\right)\to_p\sqrt{n}\left(\hat{\beta}_{\text{HO},c}-\beta\right)$.

For $\hat{\beta}_{\text{HO},c}$, based on a similar set of lemmas to Appendix B, from the same derivation as the proof of Lemma 3.1, we have

$$\sqrt{n}\left(\hat{\beta}_{\text{HO},c}-\beta\right)=\frac{\beta\sqrt{n}T_{1,n}-\beta\sqrt{n}T_{2,n}+\sqrt{n}T_{3,n}}{T_{2,n}},$$

where the denominator

$$T_{2,n}=V+O_p(n^{-1/2}),$$

and the numerator

$$
\begin{aligned}
&\beta\sqrt{n}T_{1,n} - \beta\sqrt{n}T_{2,n} + \sqrt{n}T_{3,n} \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\beta c_{i,\mathrm{HO}}\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)^2 - \beta c_{i,\mathrm{HO}}^2\left[\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)^2 + \frac{1}{J_i}\hat{\sigma}_i^2\right] + c_{i,\mathrm{HO}}\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)u_i\right] + o_p(1) \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\beta c_{i,\mathrm{HO}}\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)^2 - \beta c_{i,\mathrm{HO}}^2\left[\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)^2 + \frac{1}{J_i}\mathbb{E}\left[\sigma_i^2\right]\right] + c_{i,\mathrm{HO}}\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)u_i\right] \\
&\quad - \beta\frac{1}{n}\sum_{i=1}^{n} c_{i,\mathrm{HO}}^2\frac{\sqrt{n}}{J_i}\left(\hat{\sigma}_i^2 - \mathbb{E}\left[\sigma_i^2\right]\right) + o_p(1) \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi_i - \beta\frac{1}{n}\sum_{i=1}^{n} c_{i,\mathrm{HO}}^2\frac{\sqrt{n}}{J_i}\left(\hat{\sigma}_i^2 - \mathbb{E}\left[\sigma_i^2\right]\right) + o_p(1).
\end{aligned}
$$

For ξ_i, similar to Lemma 3.1, we have $\mathbb{E}\left[\xi_i\right] = 0$, $\mathbb{E}\left[\xi_i^2\right] \to \mathbb{E}\left[u_i^2\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)^2\right]$. However, the second term of the numerator might not disappear.

Next, following from the same arguments as Theorem 3.3, we have $\sqrt{n}\left(\hat{\beta}_{\mathrm{HO}} - \beta\right) \to_p \sqrt{n}\left(\hat{\beta}_{\mathrm{HO},c} - \beta\right)$.

Now, focusing on the bias term, by Chebyshev's inequality and (8), for any $s > 0$,

$$
\begin{aligned}
&\Pr\left(\left|\frac{1}{n}\sum_{i=1}^{n} c_{i,\mathrm{HO}}^2\frac{\sqrt{n}}{J_i}\left(\hat{\sigma}_i^2 - \mathbb{E}\left[\sigma_i^2\right]\right) - \mathbb{E}\left[c_{i,\mathrm{HO}}^2\frac{\sqrt{n}}{J_i}\left(\hat{\sigma}_i^2 - \mathbb{E}\left[\sigma_i^2\right]\right)\right]\right| > s\right) \\
&\leq \frac{\mathbb{E}\left[\frac{1}{J_i^2}\left(\hat{\sigma}_i^2 - \mathbb{E}\left[\sigma_i^2\right]\right)^2\right]}{s^2} \\
&\leq \frac{\mathbb{E}\left[\frac{1}{J_i^2}\left(\frac{K\sigma_i^4}{J_i} + \left(\sigma_i^2 - \mathbb{E}\left[\sigma_i^2\right]\right)^2\right)\right]}{s^2} \to 0.
\end{aligned}
$$

Suppose with equal probabilities, $\sigma_i^2 = 12\gamma V, J_i = \lfloor 2\sqrt{n}\rfloor$ or $\sigma_i^2 = 8\gamma V, J_i = \lfloor \frac{2}{3}\sqrt{n}\rfloor$, $\gamma > 0$. Then we have

$$
n\mathbb{E}\left[\frac{1}{J_i^2}\right] \sim \frac{1}{2}n * \frac{1}{\left(2\sqrt{n}\right)^2} + \frac{1}{2}n * \frac{1}{\left(\frac{2}{3}\sqrt{n}\right)^2} \to \frac{5}{4},
$$

$$
\begin{aligned}
&\mathbb{E}\left[c_{i,\mathrm{HO}}^2 \frac{\sqrt{n}}{J_i}\left(\hat{\sigma}_i^2 - \mathbb{E}\left[\sigma_i^2\right]\right)\right] \\
&= \mathbb{E}\left[c_{i,\mathrm{HO}}^2 \frac{\sqrt{n}}{J_i}\left(\sigma_i^2 - \mathbb{E}\left[\sigma_i^2\right]\right)\right] \\
&\sim \frac{2}{2}\left(\frac{V}{\frac{10}{2\sqrt{n}} + V}\frac{\sqrt{n}}{2\sqrt{n}} - \frac{V}{\frac{10}{\frac{2}{3}\sqrt{n}} + V}\frac{\sqrt{n}}{\frac{2}{3}\sqrt{n}}\right)\gamma V \\
&\to -\gamma V.
\end{aligned}
$$

Then by the continuous mapping theorem, in this case we have

$$
\sqrt{n}\left(\hat{\beta}_{\mathrm{HO}} - \beta\right) \to_d N(\gamma\beta, \frac{\mathbb{E}\left[u_i^2\left(\theta_i - \mathbb{E}\left[\theta_i\right]\right)^2\right]}{V^2}),
$$

which proved the first result of the theorem.

To prove the second result of the theorem, it suffices to show that if σ_i^2 and J_i are independent, then

$$
\frac{1}{n}\sum_{i=1}^{n} c_{i,\mathrm{HO}}^2 \frac{\sqrt{n}}{J_i}\left(\hat{\sigma}_i^2 - \mathbb{E}\left[\sigma_i^2\right]\right) = o_p(1).
$$

Since $c_{i,\mathrm{HO}}$ only depends on J_i, by the independence, the expectation

$$
\mathbb{E}\left[c_{i,\mathrm{HO}}^2 \frac{\sqrt{n}}{J_i}\left(\hat{\sigma}_i^2 - \mathbb{E}\left[\sigma_i^2\right]\right)\right] = 0.
$$

Then by Chebyshev's inequality,

$$
\begin{aligned}
&\Pr\left(\left|\frac{1}{n}\sum_{i=1}^{n} c_{i,\mathrm{HO}}^2 \frac{\sqrt{n}}{J_i}\left(\hat{\sigma}_i^2 - \mathbb{E}\left[\sigma_i^2\right]\right) - \mathbb{E}\left[c_{i,\mathrm{HO}}^2 \frac{\sqrt{n}}{J_i}\left(\hat{\sigma}_i^2 - \mathbb{E}\left[\sigma_i^2\right]\right)\right]\right| > s\right) \\
&\leq \frac{\mathbb{E}\left[c_{i,\mathrm{HO}}^4 \frac{1}{J_i^2}\left(\hat{\sigma}_i^2 - \mathbb{E}\left[\sigma_i^2\right]\right)^2\right]}{s^2} \\
&\leq \frac{\mathbb{E}\left[\frac{1}{J_i^2}\left(\frac{K\sigma_i^4}{J_i} + \left(\sigma_i^2 - \mathbb{E}\left[\sigma_i^2\right]\right)^2\right)\right]}{s^2} \to 0,
\end{aligned}
$$

which proves the second result of the theorem. □